

## Lecture LIV: § 19.3 (cont.) Differentiation & Integration of power series

Recall: Given  $f = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$  a power series, we have 3 options for  $f(x)$  as a function of  $x$ :

- (1)  $f$  is only defined at  $x=0$  ( $R=0$  radius of convergence)
  - (2)  $f$  is defined everywhere &  $\int_{-\infty}^{\infty}$  ( $R=\infty$  convergence is absolute everywhere)
  - (3)  $f$  is defined on  $(-R, R)$  for some max value  $0 < R < +\infty$  & perhaps also at  $x = \pm R$ . The convergence on  $(-R, R)$  is absolute & the series diverges if  $|x| > R$ .

We can find R with root / ratio test  
often

THM: Assume  $R > 0$ . Then

- (1)  $f$  is cont on  $(-R, R)$
  - (2)  $f$  ... differentiate term-by-term &  $f'(x) = a_1 + 2a_2 x + \dots = \sum_{n=1}^{\infty} n a_n x^{n-1}$
  - (3)  $f$  ... integrable  $\int f(x) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C$

In particular the two new power series have radii of convergence  $R$ . Why is this true? Let's start with

Why is this true? Let's start with (2) & (3) & show the series has  $\text{Roc} \geq R$ .  
 First approximation to (2). Say we've computed  $R$  by the Root Test:

$|x| \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$  give convergence so  $R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$

(this is close to being true always :  $\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ )

$$\text{Then } \lim_{n \rightarrow \infty} \sqrt[n]{|a_n x^{n-1}|} = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \sqrt[n]{|x|^{n-1}} = \frac{|x|}{R} \text{ exists.}$$

n<sup>th</sup> term of the series

The Root Test says if  $\frac{|x_i|}{R} < 1$  we converge & if  $\frac{|x_i|}{R} > 1$  we diverge

So the series giving  $f'$  converges <sup>as</sup> if  $|x| < R$  & diverges if  $|x| > R$

Notes: We still don't know why this series is the derivative of  $f$  !!

- We can repeat the same argument to show  $f$  is differentiable up to any order we want!

• Now, for (3) ; rewrite  $(+\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}) = \sum_{n=0}^{\infty} b_n x^n$  ( $b_0=c$ ,  $b_1=\frac{a_0}{1}$ ,  $b_2=\frac{a_1}{2}, \dots$ ) (4)

$$\text{So } |b_{n+1}| = \left| \frac{a_n}{n+1} \right| \leq |a_n|$$

Since  $\sum |a_n| |x|^n$  converges if  $|x| < R$ , by comparison,  $\sum_{n=0}^{\infty} |b_n| |x|^n$  also converges for  $|x| < R$ . So the radius of convergence of the series is at least  $R$ .

Q: How to prove (2) without <sup>the</sup> root test?

$\sum |a_n x^n|$  converges if  $|x| < R$

Given  $x$  in  $(-R, R)$ , can find  $\epsilon > 0$  so that  $0 < \tilde{x} = |x| + \epsilon < R$



So the series  $\sum_{n=0}^{\infty} |a_n (\tilde{x})^n|$  converges because  $0 < \tilde{x} \leq |x| + \epsilon < R$

Claim:  $|n x^{n-1}| \leq (|x| + \epsilon)^n$  for  $n$  large enough.

(Reason:  $\sqrt[n]{|n| |x|^{\frac{n-1}{n}}} \xrightarrow[n \rightarrow \infty]{} 1 \cdot |x| < |x| + \epsilon$   $\frac{\sqrt[n]{|n| |x|^{\frac{n-1}{n}}}}{|x|} \xrightarrow[n \rightarrow \infty]{} 1 \cdot \frac{|x|}{|x| + \epsilon}$ )

Then  $|n a_n x^{n-1}| \leq |a_n| |n x^{n-1}| \leq |a_n| (|x| + \epsilon)^n = |a_n| \tilde{x}^n$

By comparison  $\sum_{n=n_0}^{\infty} |n a_n x^{n-1}| \leq \sum_{n=n_0}^{\infty} |a_n| \tilde{x}^n$  converges.

Since the tail of the series converges absolutely, so does  $\sum_{n=1}^{\infty} |n a_n x^{n-1}|$  for  $|x| < R$   $\square$

Later this week: Rest of the proof (Appendix A15) needs absolute convergence.

Application: Can we then to find power series expansions of known functions (last time :  $e^x$ ,  $\ln(x+1)$ )

Example:  $\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + \dots$

Claim:  $\text{ROC} = 1$ .

Why? Use Ratio Test:  $\frac{(n+1)^2 |x|^{n+1}}{n^2 |x|^n} = |x| \left(\frac{n+1}{n}\right)^2 \rightarrow |x|$   
converges if  $|x| < 1$  & diverges if  $|x| > 1$ , so  $\text{ROC} = 1$ .

So  $f(x) = \sum_{n=1}^{\infty} n^2 x^n = x \underbrace{(1 + 2x^2 + 3^2 x^4 + \dots)}_{g(x)}$  is defined in  $(-1, 1)$

Note: Since derivation & integration are inverse to each other, showing  $\text{ROC} \geq R$  implies  $\text{ROC} = R$  (Otherwise f would have  $\text{ROC} > R$ .)

- So the result is also for  $g(x) = \sum_{n=0}^{\infty} (n+1)^2 x^n$ .
  - Integrate  $g$  & get a new function  $h(x) = \int_0^x g(t) dt$  also with ROC = 1  

$$h(x) = \sum_{n=0}^{\infty} \int_0^x (n+1)^2 t^n dt = \sum_{n=0}^{\infty} \frac{(n+1)^2 x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (n+1) x^{n+1},$$

$$h(x) = x + 2x^2 + 3x^3 + \dots = x(1 + \underbrace{2x + 3x^2 + \dots}_{j(x)} \text{ also has ROC = 1})$$
  - Same integration process for  $j(x)$  gives a function  $P(x)$  with ROC = 1  

$$P(x) = \int_0^x j(t) dt = \sum_{n=0}^{\infty} \int_0^x (n+1) t^n dt = \sum_{n=0}^{\infty} t^{n+1} \Big|_{t=0}^{t=x} = \sum_{n=0}^{\infty} x^{n+1}$$

$$= x(1 + x + x^2 + \dots) = \frac{x}{1-x} \quad |x| < 1$$
- Now, we reverse the process by Fundamental Thm of Calculus!

$$g(x) = P'(x) = \left(\frac{x}{1-x}\right)' = \frac{1(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$h(x) = x j(x) = \frac{x}{(1-x)^2} \quad g(x) = h'(x) = \frac{1(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4}$$

So  $f(x) = x g(x) = \frac{x(1+x)}{(1-x)^3}$  can be express as a power series in  $(-1, 1)$  ( $= \sum_{n=1}^{\infty} n^2 x^n$ ). □

Application 2: Taylor Series of  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  with ROC =  $R > 0$ .

We know  $f'(x), f''(x)$ , etc. all exist & are power series with ROC =  $R$ .

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$$

$$f''(x) = \cancel{2a_2} + 3! a_3 x + 4 \cdot 3 a_4 x^2 + 5 \cdot 4 a_5 x^3 + \dots$$

$$f'''(x) = \cancel{3! a_3} + 4 \cdot 3 \cdot 2 a_4 x + 5 \cdot 4 \cdot 3 a_5 x^2 + \dots$$

In general:  $f^{(n)}(x) = n! a_n + \text{terms containing } x \text{ as a factor}$  (series with ROC =  $R$ )

In particular,  $f^{(n)}(0) = n! a_n + 0 = n! a_n$ , gives  $a_n = \frac{f^{(n)}(0)}{n!}$

The coeffs of the power series of  $f$  come from the higher order derivatives!

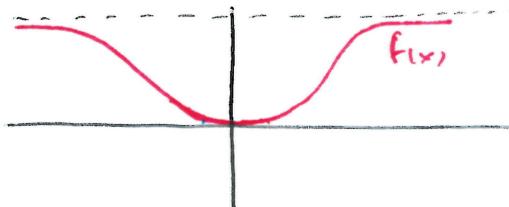
TAYLOR COEFFS
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Thm: If a function  $f$  is represented by a power series with  $\text{LOC} = R > 0$ , then this series must be the Taylor series &  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ .  
 $(f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots)$

Given a function  $f$  for which we can find  $f^{(n)}(0)$  for all  $n$ , we can write down its Taylor series  $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ .

Q: Does this series always represent  $f(x)$ ? Answer: NO

Ex:  $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$  (can compute all  $f^{(n)}(0)$  by def. & get  $f^{(n)}(0) = 0$  for all  $n$ ). So Taylor series =  $0 + 0x + 0x^2 + \dots$



$$\text{Ex: } f'(0) = \lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}} - 0}{x} = 0 \quad \text{by L'Hospital.}$$

$$f'(x) = \frac{e^{-\frac{1}{x^2}}}{x^3}$$

$$f''(0) = \lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x^4} = 0 \quad \text{etc...}$$

Easy extension: Taylor series of a function around a fixed pt  $c$  in the domain of  $f$ .

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

This works for power series centered at  $c$  as well.