

Lecture LV: § 14.9 (cont) Taylor series & Taylor's formula

Thm: If a function f can be expressed as a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ on an interval $(c-R, c+R)$ with $R > 0$, then the series is the Taylor series of f & so $a_n = \frac{f^{(n)}(c)}{n!}$.

PF/ Derivatives can be performed term-by-term.

TODAY: When can we say the Taylor series is $f(x)$? (Yesterday = not always!)

IDEA: Truncate the Taylor series & estimate the error in approximation.

Write $f(x) = \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n}_{\text{degree } n \text{ Taylor Polynomial} = T_n(x)} + \underbrace{R_n(x)}_{\text{Remainder}}$

Fact: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ converges to $f(x)$ if and only if $|R_n(x)| \xrightarrow{n \rightarrow \infty} 0$ for every x in $(-R, R)$

This is only useful if we can give a formula for $R_n(x)$.

(Lagrange) Remainder Formula
centered at 0

$$R_n(x) = \frac{f^{(n+1)}(b)}{(n+1)!} x^{n+1} \quad \text{for some } b \text{ between } 0 \text{ \& } x.$$

§1 Applications:

Application 1: Show $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ Know by Ratio Test that ROC = ∞ .

• Taylor series of e^x at $x=0$: $f(x) = e^x$ so $f^{(n)}(0) = 1$ for all n .

• Remainder: $|R_n(x)| = \left| \frac{e^b}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1} e^x}{(n+1)!} \xrightarrow{n \rightarrow \infty} 0$ (x is fixed)

So we know that e^x can be expressed by its Taylor series (we already knew the identity!)

Application 2: Taylor series for $\sin(x)$.

$f(x) = \sin x$ $f(0) = 0$

$f'(x) = \cos x$ $f'(0) = 1$

$f''(x) = -\sin x$ $f''(0) = 0$

$f'''(x) = -\cos x$ $f'''(0) = -1$

$f^{(4)}(x) = \sin x$ $f^{(4)}(0) = 0$

& repeats from then on: So the Taylor series

of $f(x)$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

ONLY odd powers & $\sin(x)$ is an odd function! (no coincidence!)

Ratio Test says $\frac{\left| \frac{x^{2n+3}}{(2n+3)!} \right|}{\left| \frac{x^{2n+1}}{(2n+1)!} \right|} = \frac{|x|^2}{(2n+2)(2n+3)} \xrightarrow{n \rightarrow \infty} 0$ absolutely converges for all x

ROC = $+\infty$

We want to show Taylor series converges to f .

Use Remainder formula:

$$|R_n(x)| = \frac{|x|^{n+1}}{(n+1)!} \left| f^{(n+1)}(b) \right| \leq \frac{|x|^{n+1}}{(n+1)!} \xrightarrow{n \rightarrow \infty} 0$$

$\pm \sin(b)$ or $\pm \cos(b)$

b/c series of e^x converges for all

So $\sin(x)$ equals its Taylor series & ROC = $+\infty$.

Application 3: $\cos(x) = (\sin(x))'$ & since $\sin(x)$ is a power series, we can take derivatives term by term.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

& ROC = $+\infty$
only EVEN powers & $\cos(x)$ is even func.

Also we can see the (RHS) is the Taylor series of $\cos(x)$ by Thm.

§2 Other manipulations

① Substitution: From $e^{\tilde{x}} = \sum_{n=0}^{\infty} \frac{\tilde{x}^n}{n!}$ to e^{-x^2}

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

for all x . So it must be the Taylor series of e^{-x^2} (ROC = $+\infty$)

Set $\tilde{x} = -x^2$
OK because ROC = $+\infty$

② Integration of e^{-x^2} can be approximated!

$$\int_0^1 e^{-x^2} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n! (2n+1)} \Big|_0^1$$

Integrate term by term

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (2n+1)} = \frac{1}{3} - \frac{1}{2! \cdot 5} + \frac{1}{3! \cdot 7} - \frac{1}{4! \cdot 9} + \dots$$

Q: Other centers? $R_n(x) = \frac{f^{(n+1)}(b)}{(n+1)!} (x-c)^{n+1}$ for b between c & x .

Taylor series converges to f if and only if $|R_n(x)| \xrightarrow{n \rightarrow \infty} 0$.

Example: Expansion of $f(x) = \sin(x)$ as a Taylor series about $x = \frac{\pi}{2}$

$f(x) = \sin(x)$	$f(\frac{\pi}{2}) = 1$
$f'(x) = \cos(x)$	$f'(\frac{\pi}{2}) = 0$
$f''(x) = -\sin(x)$	$f''(\frac{\pi}{2}) = -1$
$f'''(x) = -\cos(x)$	$f'''(\frac{\pi}{2}) = 0$
$f^{(4)}(x) = \sin(x)$	$f^{(4)}(\frac{\pi}{2}) = 1$

The Taylor series is then:

$$1 - \frac{(x - \frac{\pi}{2})^2}{2!} + \frac{(x - \frac{\pi}{2})^4}{4!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x - \frac{\pi}{2})^{2n}}{(2n)!}$$


 $\sin(x)$ is odd about $x = \frac{\pi}{2}$ & only odd exp

Check that it converges to $f(x)$. We know ROC = $+\infty$ because $\frac{|x - \frac{\pi}{2}|^{2n+2}}{(2n+2)!} \rightarrow 0$ as $n \rightarrow \infty$ for any x .

Remainder: $|R_n(x)| = \left| \frac{f^{(n+1)}(b)}{(n+1)!} (x - \frac{\pi}{2})^{n+1} \right| \leq \frac{|x - \frac{\pi}{2}|^{n+1}}{(n+1)!} \rightarrow 0$ as $n \rightarrow \infty$ for any x .

Note: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x - \frac{\pi}{2})^{2n}}{(2n)!} = \cos(x - \frac{\pi}{2})$

substitute in Taylor series of $\cos(x)$ at $x - \frac{\pi}{2}$

Proof of Lagrange Remainder Formula:

Pick 0 = center of Taylor series & write $R_n(x) = S_n(x)(x-0)^{n+1}$ for $x \neq 0$
 (This is the Tail of Taylor series)

Now: fix x & define a new function $F(t)$ for $\begin{cases} 0 \leq t \leq x & \text{if } x > 0 \\ x \leq t \leq 0 & \text{if } x < 0 \end{cases}$

$$F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \dots - \frac{f^{(n)}(t)}{n!}(x-t)^n - S_n(x)(x-t)^{n+1}$$

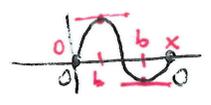
deg n Taylor polynomial of f at t .

$F(0) = f(x) - f(0) - f'(0)(x) - \frac{f''(0)}{2!}x^2 - \dots - \frac{f^{(n)}(0)}{n!}x^n - S_n(x)x^{n+1} = 0$
 by def of $R_n(x) = f(x) - \text{deg } n \text{ Taylor poly at } 0$.

$F(x) = 0$ (all terms = 0 when $t = x$).

F is cont on $[0, x]$ because f is infinitely differentiable so all $f^{(n)}(t)$ are cont & $(x-t)^n$ is continuous in t .

F is diff'ble on $(0, x)$ for the same reasons.



By the Mean value Theorem, can find $0 < b < x$ with $F'(b) = 0$

Write $F'(t) = -f'(t) - (f''(t)(x-t) + f'(t)(-1)) - (f'''(t)(x-t)^2 + \frac{f''(t)}{2!} 2(x-t)(-1)) - \frac{f^{(4)}(t)}{3!} (x-t)^3 - \frac{f^{(4)}(t)}{3!} 3(x-t)^2 \dots - (f^{(n+1)}(t)(x-t)^n + \frac{f^{(n)}(t)}{n!} n(x-t)^{n-1}(-1)) + S_n(x)(x-t)^n$

$= (-f'(t) + f'(t)) + (-f''(t)(x-t) + f''(t)(x-t)) + (-\frac{f'''(t)}{2!} (x-t)^2 + \frac{f'''(t)}{2!} 2(x-t)(-1)) + 0 + \dots + 0 - \frac{f^{(n+1)}(t)}{n!} (x-t)^n + S_n(x)(n+1)(x-t)^n$

So $0 = F'(b) = -\frac{f^{(n+1)}(b)}{n!} (x-b)^n + S_n(x)(n+1)(x-b)^n$

$0 = (n+1)(x-b)^n \left(\frac{-f^{(n+1)}(b)}{(n+1)!} + S_n(x) \right)$ $x = b \neq 0$

So $\frac{f^{(n+1)}(b)}{(n+1)!} = S_n(x)$ & $R_n(x) \sim \frac{f^{(n+1)}(b)}{(n+1)!} x^{n+1}$ \square

The formula for another center c follows by changing variables $z = x - c$

$f(x) = g(z+c)$ $z = x - c$ $g(b) = f(b+c)$
 Taylor series around c \hookrightarrow Taylor series centered at 0 $0 \leq b \leq z$ in between c & x

$R_n(x) \approx f = \tilde{R}_n(x) \approx g = \frac{g^{(n+1)}(b)}{(n+1)!} (z)^{n+1} = \frac{f^{(n+1)}(b)}{(n+1)!} (x-c)^{n+1}$
 $\hookrightarrow \tilde{b}$ between c & x