

§1) Power series & their continuity & Uniform convergence

Fix a power series  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  with radius of convergence  $R > 0$

$$f(x) = S_n(x) + R_n(x)$$

where  $S_n(x) = a_0 + a_1 x + \dots + a_n x^n$  ( $n^{\text{th}}$  partial sum)  
 $R_n(x) = f(x) - S_n(x) = a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots = \text{Tail.}$

If  $x$  in  $(-R, R)$  we know  $S_n(x) \xrightarrow{n \rightarrow \infty} f(x)$  so  $R_n(x) \xrightarrow{n \rightarrow \infty} 0$ ,  
 meaning that for every  $\epsilon > 0$ , we can find  $n_0 = n_0(\epsilon, x)$  where  
 $|R_n(x)| < \epsilon$  if  $n \geq n_0$ . [Warning:  $n_0$  depends on  $\epsilon$  &  $x$  (in principle!)]

Prop:  $R_n(x) \xrightarrow{n \rightarrow \infty} 0$  uniformly if  $x$  in  $[-R', R']$  for any  $0 < R' < R$

(meaning that  $n_0 = n_0(\epsilon)$  so independent of  $x$ )

Why? Since  $|x| \leq R' < R$ , we get  $|R_n(x)| = \left| \sum_{k=n+1}^{\infty} a_k x^k \right| \leq \sum_{k=n+1}^{\infty} |a_k| |x|^k$   
 $\leq \sum_{k=n+1}^{\infty} |a_k| (R')^k = \text{Tail of } \sum_{k=0}^{\infty} |a_k| |R'|^k$

Since  $f(x)$  converges absolutely because  $0 < R' < R$ , we can find  $n_0$  where  
 $\sum_{k=n_0+1}^{\infty} |a_k| (R')^k < \epsilon$  so  $|R_n(x)| \leq \sum_{k=n+1}^{\infty} |a_k| (R')^k \leq \sum_{k=n_0+1}^{\infty} |a_k| (R')^k < \epsilon$   
 if  $n \geq n_0$ .

• Continuity follows directly from uniform convergence!

Prop:  $f(x)$  is continuous in  $(-R, R)$ .

Proof: given  $x_0$  in  $(-R, R)$ , want to show  
 $\lim_{x \rightarrow x_0} f(x) = f(x_0)$  Q: What  $n$  to pick?



Pick any  $\epsilon > 0$  & write  $f(x) = S_n(x) + R_n(x)$  (with  $R_n(x)$  boxed and labeled 'tail')

Pick  $\delta_1 > 0$  with  $-R < x_0 - \delta_1 < x_0 + \delta_1 < R$  ( $\delta_1 = \frac{R - |x_0|}{2}$  will do)

Call  $R' = \max\{|x_0 + \delta_1|, |x_0 - \delta_1|\} < R$  (by construction).

We can find  $n_0 = n_0(\epsilon)$  where  $|R_n(x)| < \frac{\epsilon}{3}$  if  $x \in (-R', R')$

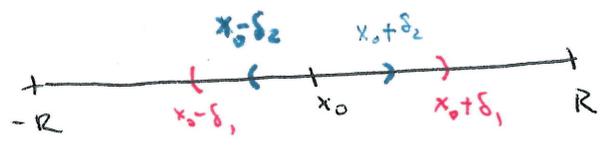
(by uniform convergence of  $R_n$ .)

In part  $|R_{n_0}(x_0)| < \frac{\epsilon}{3}$  &  $|R_{n_0}(x)| < \frac{\epsilon}{3}$  if  $x \in (x_0 - \delta_1, x_0 + \delta_1)$   
 Pick  $n = n_0$  &  $f(x) = S_{n_0}(x) + R_{n_0}(x)$

Now  $|f(x) - f(x_0)| = |S_n(x) + R_n(x) - (S_n(x_0) + R_n(x_0))|$   
 $= |S_n(x) - S_n(x_0) + R_n(x) - R_n(x_0)|$   
 $\leq |S_n(x) - S_n(x_0)| + |R_n(x)| + |R_n(x_0)|$

But  $S_{n_0}(x) = a_0 + a_1x + \dots + a_{n_0}x^{n_0}$  is a polynomial, so it's continuous. We can find  $\delta_2 > 0$  for which  $|S_{n_0}(x) - S_{n_0}(x_0)| < \frac{\epsilon}{3}$  if  $x$  in  $(x_0 - \delta_2, x_0 + \delta_2)$ .

Pick  $\delta = \min\{\delta_1, \delta_2\}$



Then  $|f(x) - f(x_0)| \leq |S_{n_0}(x) - S_{n_0}(x_0)| + |R_{n_0}(x)| + |R_{n_0}(x_0)|$   
 $|f(x) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$   
 if  $x$  in  $(x_0 - \delta, x_0 + \delta)$ . So  $\lim_{x \rightarrow x_0} |f(x) - f(x_0)| = 0$ .

§2 Term-by-term integration:

Assume  $\text{ROC} = R > 0$ . Want to show  $\int_c^d \sum_n a_n x^n dx = \sum_n \int_c^d a_n x^n dx$   
 $-R < c < d < R$

How? Write  $f(x) = \sum_{(x), n} a_n x^n = S_n(x) + R_n(x)$  &  $f, S_n, R_n$  are  
 cont poly cont. so cont! continuous & we can integrate them!

$S_{n_0}(x)$  is a polynomial so we can integrate term-by-term.

$\int_c^d f(x) dx = \int_c^d S_n(x) dx + \int_c^d R_n(x) dx$   
 $= \sum_{k=0}^n \int_c^d a_k x^k dx + \int_c^d R_n(x) dx \stackrel{?}{=} \sum_{k=0}^{\infty} \int_c^d a_k x^k dx$

Need to show  $|\int_c^d R_n(x) dx| = |\text{tail of } \sum_{k=0}^{\infty} \int_c^d a_k x^k dx| \xrightarrow{n \rightarrow \infty} 0$   
 the sequence

Use uniform convergence of  $R_n$ ! (this is a number) for each  $n$ . Fix  $\epsilon > 0$ , & want to find  $n_0$  with  $|\int_c^d R_n(x) dx| < \epsilon$  if  $n \geq n_0$ .

Pick  $R' = \max\{|c|, |d|\} < R$  so  $c, d$  in  $(-R', R')$  &  $|R_n(x)| < \frac{\epsilon}{d-c}$  if  $n \geq n_0$  to any  $x$  in  $(-R', R')$

by unif conv.  
 Then  $|\int_c^d R_n(x) dx| \leq \int_c^d |R_n(x)| dx < \int_c^d \epsilon dx = \epsilon(d-c) = \epsilon$   
 $\downarrow n \geq n_0$

Consequence  $\int_0^x \sum_{n=0}^{\infty} a_n t^n dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$  for  $-R < x < R$

### § 3 Differentiation term by term:

Recall (Lecture LIV) If  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  has ROC  $R > 0$ , then

$g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$  also has ROC at least  $R$ .

Missing point: this series is  $f'$ !

Since  $g(x)$  is continuous in  $(-R, R)$  & we can integrate term-by-term, we get

$$\int_0^x g(t) dt = \sum_{n=1}^{\infty} \int_0^x n a_n t^{n-1} dt = \sum_{n=1}^{\infty} a_n t^n \Big|_0^x = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$$

So  $f(x) = a_0 + \underbrace{\int_0^x g(t) dt}_{\text{differentiable!}}$  is differentiable & by F.T.C. we get  $f'(x) = g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ .

### § 4 Application to diff'l equations:

Ex 1  $y' = y$

1. Propose a solution  $\sum_{n=0}^{\infty} a_n x^n$  with ROC  $> 0$

2. Use the eqn to find a relations among  $a_0, a_1, \dots$

(Eg  $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$  so  $\begin{matrix} 1 a_1 = a_0 \\ 2 a_2 = a_1 = a_0 \\ 3 a_3 = a_2 = \frac{a_0}{2} \end{matrix}$ )  
we get  $a_n = \frac{a_0}{n!}$ )

3. Write down the series & check if ROC is  $> 0$ .

(Eg: get  $a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{3!} x^3 + \dots = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$   
Use Ratio Test to get ROC =  $\infty$  ✓)

4. If we are lucky, we can recognize the power series as a function we know (Eg =  $a_0 e^x$ ).

Ex 2:  $y'' + y = 0$ .

The solution  $y = \sum_{n=0}^{\infty} a_n x^n$  satisfies the recursive

relation:  $\sum_{n=0}^{\infty} ((n+1)(n+2) a_{n+2} + a_n) x^n = 0$

so  $a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$

Values of  $a_0$  determines  $a_2, a_4, a_6, \dots$

Value of  $a_1$  ———  $a_3, a_5, a_7, \dots$

Even coeffs  $a_{2n} = \frac{(-1)^n a_0}{(2n)!}$   $\leadsto$  give a new abs convergent series with ROC =  $\infty$

Odd coeffs  $a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}$   $\leadsto$  with ROC =  $\infty$

So  $y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$  (can recognize purely by abs conv.)  
 $= a_0 \cos(x) + a_1 \sin(x)$

Ex 3 Bessel's eqn:  $xy'' + y' + xy = 0$

Propose  $y = \sum_{n=0}^{\infty} a_n x^n$   $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$   $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$a_1 + \sum_{n=2}^{\infty} [(n-1)n a_n + n a_n + a_{n-2}] x^{n-1} = 0$

gives  $a_1 = 0$   
 $(n-1)n a_n + n a_n + a_{n-2} = n^2 a_n + a_{n-2} = 0 \leadsto a_n = -\frac{a_{n-2}}{n^2}$  for  $n \geq 2$

Soln:  $a_{2n} = a_0 \frac{(-1)^n}{2^2 4^2 \dots (2n)^2} = \frac{a_0 (-1)^n}{2^{2n} (n!)^2}$   
 odd coeffs = 0.  
 even "  $a_{2n} = \frac{(-1)^n a_0}{(2n!)^2}$

$y_{(x)} = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$  check ROC =  $+\infty$  by Ratio Test.  
 Bessel function  $J_0(x)$  of order 0.