

## Lecture LVIII: § 14.7 Operations on power series

GOAL: Find the Taylor series of functions  $f(x)$ , that are power series with  $\text{ROC} > 0$ .  
 So we know  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

- Want to avoid the computation of all  $f^{(n)}(x)$ !
- By uniqueness if we compute the power series expansion of  $f$  it

MUST equal its Taylor series, so we can use algebraic manipulations.

Example 1: Substitution of one series in other one:  $f(g(x))$  for  $|x| < 1 = \text{ROC}$  {① Substitution  
② Products  
③ Long Division}

$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \dots$

We replace  $x$  by  $x^4 = g(x)$  in both sides & if  $|x| < 1$  then  $|x^4| < 1$ .  
 $(|g(x)| < \text{ROC})$

(so this is an allowed operation!)

$$f(g(x)) = f(x^4) = \frac{1}{1-x^4} = 1 + x^4 + x^8 + \dots = \sum_{k=0}^{\infty} x^{4k}$$

$$= \frac{b_0 + b_1 x^5}{1-x^4} = x^5 + x^9 + x^{13} + \dots = \sum_{k=0}^{\infty} x^{4k+5}$$

valid on  $(-1, 1)$

So  $\frac{g(n)}{n!} = \begin{cases} 0 & n \text{ not divisible by } 4 \\ 1 & \text{if } n = 4k \end{cases}$

$$\& \frac{h_2(n)}{n!} = \begin{cases} 0 & \text{if } n \neq 4k+5 \\ 1 & \text{if } n = 4k+5 \text{ for some } k \end{cases}$$

In general:  $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$  &  $g(x) = b_0 + b_1 x + b_2 x^2 + \dots$

$$f(g(x)) = a_0 + a_1 g(x) + a_2 (g(x))^2 + \dots = a_0 + a_1(b_0 + b_1 x + b_2 x^2 + \dots) + a_2(b_0^2 + b_0 b_1 x + b_1^2 x^2 + \dots) + \dots$$

→ Need to multiply two series & show  $f(g(x))$  converges if  $|g(x)| < R$

Example 2: Product of 2 power series

$$f(x) = e^x, \quad g(x) = \sin(x)$$

Q: Is  $f(x)g(x)$  also a power series?

A: YES &  $\text{ROC} = \min \{\text{ROC}(f), \text{ROC}(g)\}$

Idea: Use distribution Laws!

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

If  $f(x)g(x) = \sum_{n=0}^{\infty} a_n x^n$  we can get  $a_n x^n$  from multiplying the 2 & collecting common terms for each  $x^n$ .

$$\begin{aligned}
 f(x)g(x) &= \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \\
 &\quad + \left( x^2 - \frac{x^4}{3!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots \right) \\
 &\quad + \left( \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^7}{8!5!} - \frac{x^9}{2\cdot7!} + \dots \right) \\
 &\quad + \left( \frac{x^4}{3!} - \frac{x^6}{3!3!} + \frac{x^8}{3!5!} - \frac{x^{10}}{3!7!} + \dots \right)
 \end{aligned}$$

$f(x)$   
 $+ x \cdot g(x)$   
 $\frac{x^2}{2} g(x)$   
 $\frac{x^3}{3!} g(x)$

Coeff of  $x$  : 1

" "  $x^2$  : 1

Coeff of  $x^3$  :  $-\frac{1}{3!} + \frac{1}{2} = \frac{1}{3}$

Coeff of  $x^4$  :  $-\frac{1}{3!} + \frac{1}{3!} = 0$

(complicated formula)

$$So e^x \ln(x) = x + x^2 + \frac{1}{3}x^3 + \dots \quad \text{with ROC} = \infty.$$

Another example :  $f(x) = \ln(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots)$   $|x| < 1$

$$g(x) = \frac{1}{x-1} = - (1+x+x^2+\dots) \quad |x| < 1$$

$$\begin{aligned}
 f(x)g(x) &= \left( x + x^2 + x^3 + x^4 + \dots \right) \\
 &\quad + \left( -\frac{x^2}{2} - \frac{x^3}{2} - \frac{x^4}{2} - \dots \right) = \sum_{n=1}^{\infty} \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} \right) x^n \\
 &\quad \frac{x^3}{3} + \frac{x^4}{3} + \dots \\
 &\quad \dots \\
 &\hline
 &\quad x + \left( 1 + \frac{1}{2} \right) x^2 + \left( 1 + \frac{1}{2} + \frac{1}{3} \right) x^3 + \dots
 \end{aligned}$$

↑ easy formula

Q: Is this process correct?

A: We are rearranging a series!

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{with ROC } R_1 > 0$$

$$g(x) = \sum_{m=0}^{\infty} b_m x^m \quad " " \quad R_2 > 0$$

Term-by-Term multiplication :

$$a_0 g(x) = a_0 b_0 + a_0 b_1 x + a_0 b_2 x^2 + \dots$$

$$a_1 x g(x) = a_1 b_0 x + a_1 b_1 x^2 + \dots$$

$$a_2 x^2 g(x) = a_2 b_0 x^2 + a_2 b_1 x^3 + \dots$$

Then, we add by columns!

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \dots$$

We propose  $f(x)g(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n (a_k b_{n-k}) \right) x^n \quad (*)$

Claim: This series converges absolutely if  $|x| < R = \min\{R_1, R_2\}$

Why? Take partial sums of  $f(x)$  &  $g(x)$ :

$$S_n = a_0 + a_1 x + \dots + a_n x^n$$

$$T_n = b_0 + b_1 x + \dots + b_n x^n$$

$$S_n T_n = \sum_{p=0}^{2n} \left( \sum_{k=0}^p (a_k b_{p-k}) \right) x^p$$

Arrange them as

$a_0 b_0$	$a_0 b_1$	$a_0 b_2$	$a_0 b_3$
$a_1 b_0$	$a_1 b_1$	$a_1 b_2$	
$a_2 b_0$	$a_2 b_1$	$a_2 b_2$	
$a_3 b_0$	$a_3 b_1$		
$a_4 b_0$			

① Sum the L's  
 ② Sum along antidiag is (missing terms) partial sum of  $(*)$

By absolute convergence

we can rearrange the series in ANY way we want & get the same result for both ways of adding terms.

①  $S_n T_n \xrightarrow{n \rightarrow \infty} f(x)g(x)$

② Series  $(*)$ .

rows:  $a_i x^i g(x)$

columns:  $b_j x^j f(x)$

Really need abs. convergence in  $(*)$ !

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n+1}} = 1 - \frac{x}{\sqrt{2}} + \frac{x^2}{\sqrt{3}} - \dots$$

not abs. conv. ROC = 1.  
 only conditional convergence.

Series looks like:  $\sum_{n=0}^{\infty} \left( \sum_{k=0}^n \frac{(-1)^n}{\sqrt{k+1}} \frac{1}{\sqrt{n-k+1}} \right) x^n$  It diverges to  $x=1$

Example 3: Long division of power series  $\frac{f(x)}{g(x)}$  when  $g(0) = 1$

gives a new power series with radius of convergence  $> 0$  (need to avoid zeros of  $g(x)$ !)

Ex:  $\tan(x) = \frac{\sin x}{\cos x}$  will have a power series expansion with radius of convergence  $R = \frac{\pi}{2}$  ( $\cos \frac{\pi}{2} = 0 = \cos -\frac{\pi}{2}$  gives no zeros in  $(-\frac{\pi}{2}, \frac{\pi}{2})$ )

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \\ &\quad - \boxed{x} + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \\ &\quad - \frac{x - \frac{x^3}{2}}{1} + \frac{x^5}{24} - \dots \\ &\quad \boxed{\frac{1}{3}x^3} - \frac{1}{30}x^5 + \dots \\ &\quad \frac{1}{3}x^3 - \frac{1}{6}x^5 + \dots \\ &\quad \frac{2}{15}x^5 + \dots \end{aligned}$$

Key:  $\cos x = ① + \text{terms with } x.$

$\Rightarrow$  We can write  $\frac{1}{\cos x}$  as a power series (Appendix A16)

Prop: If  $\sum_{n=0}^{\infty} b_n x^n$  has  $b_0 \neq 0$  & ROC > 0, then  $\sum_{n=0}^{\infty} b_n x^n$  has a power series expansion with positive ROC.

Why? (1) Propn  $\frac{1}{\sum b_n x^n} = \sum_{n=0}^{\infty} c_n x^n$  & find  $c_n$  by recursion.

$$\begin{aligned} 1 &= \left( \sum_{n=0}^{\infty} b_n x^n \right) \left( \sum_{n=0}^{\infty} c_n x^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n b_{n-k} c_k \right) x^n \\ &= b_0 c_0 + (b_0 c_1 + b_1 c_0) x + (b_0 c_2 + b_1 c_1 + b_2 c_0) x^2 + \dots \end{aligned}$$

$$\Rightarrow b_0 c_0 = 1 \quad \& \quad b_0 \neq 0 \Rightarrow \boxed{c_0 = \frac{1}{b_0}}$$

$$\begin{aligned} \cdot b_0 c_1 + b_1 c_0 &= 0 \quad \& \quad b_0 \neq 0 \Rightarrow c_1 = -\frac{b_1}{b_0} c_0 = \boxed{-\frac{b_1}{b_0^2}}. \\ \cdot b_0 c_2 + b_1 c_1 + b_2 c_0 &= 0 \quad \Rightarrow c_2 = -\frac{1}{b_0} (b_1 c_1 + b_2 c_0) \end{aligned}$$

(2) Show the series  $\sum_{n=0}^{\infty} c_n x^n$  has positive ROC. known!

- $c_n = -\sum_{k=0}^{n-1} \frac{b_{n-k} c_k}{b_0}$  for all  $n$
- Can assume  $b_0 = 1 \neq 0$  (otherwise  $1 = b_0 \left( \sum_{n=0}^{\infty} \frac{b_n}{b_0} x^n \right) + \frac{1}{b_0} \left( \sum_{n=0}^{\infty} (c_n b_0) x^n \right)$ )

- Since  $\sum_{n=0}^{\infty} b_n x^n$  has ROC  $R > 0$ , pick  $0 < r < R$  & get  
 $\sum_{n=0}^{\infty} |b_n|r^n$  converges, so  $|b_n|r^n \xrightarrow{n \rightarrow \infty} 0$

In particular, the sequence  $\{|b_n|r^n\}_n$  is bounded & we can find  $K \geq 1$  with  $|b_n|r^n \leq K$  for all  $n$

$$|b_n| \leq \frac{K}{r^n}$$

- $|c_0| = 1 \leq K$

$$|c_1| = |b_1 c_0| = |b_1| \leq \frac{K}{r}$$

$$|c_2| = |b_1 c_1 + b_2 c_0| \leq |b_1| |c_1| + |b_2| \leq \frac{K}{r} \frac{K}{r} + \frac{K}{r^2} K = \frac{2K^3}{r^2}$$

$$\begin{aligned} |c_3| &= |b_1 c_2 + b_2 c_1 + b_3 c_0| \leq |b_1 c_2| + |b_2 c_1| + |b_3 c_0| \\ &\leq \frac{K}{r} \frac{2K^3}{r^2} + \frac{K}{r^2} \frac{K}{r} + \frac{K}{r^3} K \\ &\stackrel{K_2, 1}{\leq} \frac{2K^3}{r^3} + \frac{K^3}{r^3} + \frac{K^3}{r^3} = \frac{4K^3}{r^3} = \frac{2^3 K^3}{r^3} \end{aligned}$$

In general

$$\begin{aligned} |c_n| &\leq |b_1 c_{n-1}| + |b_2 c_{n-2}| + \dots + |b_{n-1}| |c_1| + |b_n| \\ &\leq \frac{K}{r} \frac{2^{n-2} K^{n-1}}{r^{n-1}} + \frac{K}{r^2} \frac{2^{n-3} K^{n-2}}{r^{n-2}} + \dots + \frac{K}{r^{n-1}} \frac{K}{r} + \frac{K}{r^n} K \\ &\stackrel{K \geq 1}{\leq} (2^{n-2} + 2^{n-3} + \dots + 1 + 1) \frac{K^n}{r^n} = 2^{n-1} \frac{K^n}{r^n} \leq 2^n \frac{K^n}{r^n} \end{aligned}$$

$$\text{So } \sum_{n=0}^{\infty} |c_n| x^n \leq \sum_{n=0}^{\infty} 2^{n-1} \frac{K^n}{r^n} |x^n| \leq \sum_{n=0}^{\infty} \frac{2^n K^n}{r^n} |x|^n = \sum_{n=0}^{\infty} \left| \frac{2Kx}{r} \right|^n$$

& this converges if  $\left| \frac{2Kx}{r} \right| < 1$  that is  $|x| < \frac{r}{2K}$ .

So the ROC of the series  $\sum_{n=0}^{\infty} c_n x^n$  is at least  $\frac{r}{2K} > 0$ .