

# Lecture LIX : §14.8 Complex numbers & Euler's formula

## §1 Motivation:

① Algebraic: Wants to extend  $\mathbb{R}$  so that every polynomial has a root.

( $x^2 + 1$  has no root in  $\mathbb{R}$ , so we create  $i = \sqrt{-1}$  and add it to  $\mathbb{R}$ )

$$\text{Now } (x^2 + 1) = (x - i)(x + i)$$

② Geometric: Want to define multiplication of points in  $\mathbb{R}^2 = \mathbb{C}$

## §2 Definition:

A complex number  $z$  is given by a pair of real numbers  $(a, b)$  in  $\mathbb{R}^2$ , written as  $z = a + ib$  ( $i^2 = -1$ ). We write:  $\begin{cases} a = \text{Real part of } z = \text{Re}(z) \\ b = \text{Imaginary part of } z = \text{Im}(z) \end{cases}$  Call  $\mathbb{C}$  the set of all complex numbers.

(1) Addition is component wise

$$(a+ib) + (c+id) = (a+c) + i(b+d)$$

(2) Multiplication is determined by distributive laws &  $i^2 = -1$ .  
[pretend  $i$  is a variable]

$$\begin{aligned} z = a+ib \\ w = c+id \end{aligned} \Rightarrow zw = ac + iad + ibc + i^2 bd = \boxed{(ac-bd) + i(ad+bc)}$$

new Re &  
Im parts

$$\text{Eg: } z = 1+i \quad w = 2+3i \quad \Rightarrow zw = (2-3) + i(3+2) = -1+5i$$

$$z^2 = (1+i)^2 = (1-1) + i(1+1) = 2i$$

Properties:

$$(1) zw = wz$$

$$(2) z(w+v) = zw + zv$$

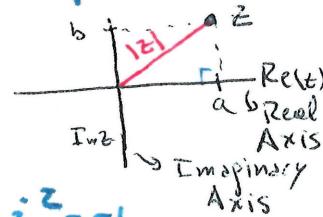
(3)  $\mathbb{R}$  line in  $\mathbb{C}$   $a = a+0i$ . & addition & multiplication

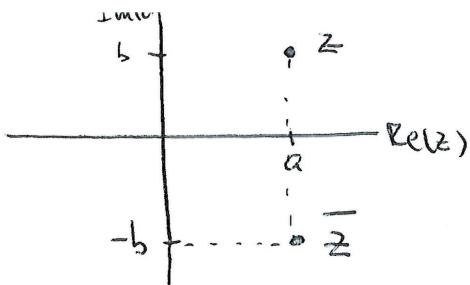
for  $a \in \mathbb{R}$  agree w/ add & mult of pts  $(a+0i)$  though in  $\mathbb{C}$ .

[We extended  $\mathbb{R}$  by adding more numbers without breaking its structure]

Def 2: The modulus of a complex number is given by  $|a+bi| = \sqrt{a^2+b^2}$

Given  $z = a+ib$ , we call  $\bar{z} = a-ib$  the complex conjugate





$$\text{Note} \cdot \overline{z+w} = \overline{z} + \overline{w}$$

$$\cdot \overline{zw} = \overline{z} \cdot \overline{w} \quad (\text{so } \overline{z^n} = (\overline{z})^n \text{ for all } n \geq 1)$$

$$\text{keys. } z \cdot \overline{z} = |z|^2$$

$$\text{why? } (a+bi)(a-bi) = (a^2+b^2) + (ab-ba)i = a^2+b^2 = |z|^2.$$

$$\cdot z + \overline{z} = 2a = 2\operatorname{Re}(z)$$

$$z - \overline{z} = 2bi = 2\operatorname{Im}(z)i$$

• By def.  $\frac{z}{a+bi} \neq 0$  if either  $a \neq 0$  or  $b \neq 0$ , that is if and only if  $|z| \neq 0$

• So from  $z \cdot \overline{z} = |z|^2$ , we get  $z \cdot \frac{\overline{z}}{|z|^2} = 1$  so  $z^{-1} = \frac{\overline{z}}{|z|^2}$

Example:  $z = 1+i \quad |z| = \sqrt{1+1} = \sqrt{2}$

$$z^{-1} = \frac{1-i}{(\sqrt{2})^2} = \frac{1}{2} - i \frac{1}{2}$$

multiplication  
inverse!

Can use this to write any quotient  $\frac{a+bi}{c+di}$  as a complex number!

$$\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2}i$$

Example:  $\frac{2+i}{3-i} = \frac{6-1}{10} + \frac{(3-(-2))i}{10} = \frac{5+5i}{10} = \boxed{\frac{1}{2} + i \frac{1}{2}}$

Fundamental Thm of Algebra: Every polynomial with complex coefficients has a root in  $\mathbb{C}$ .

Example quadratic polynomials  $ax^2+bx+c$  with  $a \neq 0$   $a, b, c$  in  $\mathbb{R}$ .

$$\text{Complete squares: } a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

so if  $x$  is a root, then

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \Rightarrow \text{need } b^2 - 4ac \text{ to be } \geq 0 \text{ for a real root.}$$

$$\text{otherwise } b^2 - 4ac < 0 \quad \& \quad b^2 - 4ac = -1 \quad \underline{(4ac - b^2)}$$

$$\therefore x = \frac{-b}{2a} \pm i \frac{\sqrt{4ac - b^2}}{2a} \quad \text{we have 2 non-real complex roots!}$$

Ex:  $x^2 + x + 5 = 0$  has roots:  $\frac{-1 \pm \sqrt{19}i}{2}$ .

Ex: Find  $\sqrt{2+i} = a+ib$   $(a+ib)^2 = (a^2 - b^2) + 2abi = 2+i$   $\begin{cases} a^2 - b^2 = 2 \\ 2ab = 1 \end{cases}$

$$\begin{cases} a^2 = 2 + b^2 \\ 2ab = 1 \end{cases}$$

$$\text{get } a^2 - b^2 = \frac{1}{4b^2} - b^2 = 2 \Rightarrow 1 - 4b^2 + 8b^2 = 0$$

$$\text{so } b^2 = \frac{-8 \pm \sqrt{8^2 + 4 \cdot 4}}{8} = \frac{-8 \pm 4\sqrt{5}}{8}$$

$b$  is real so can only have  $b^2 \geq 0$ , so + sign is only valid.  $b^2 = \frac{\sqrt{5}}{2} - 1$

$$b = \pm \sqrt{\frac{\sqrt{5}}{2} - 1} \quad \& \quad a = \frac{1}{\pm 2\sqrt{\frac{\sqrt{5}}{2} - 1}}$$

### § 3 Polar coordinates & Euler's formula

$$z = a + ib \quad |z| = \sqrt{a^2 + b^2}$$

$$\Rightarrow z = |z| \cos \theta + i |z| \sin \theta = |z| (\cos \theta + i \sin \theta)$$

$$\text{Example} \quad 1+i = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

Q: Can we add, multiply in polar words?

NO      YES

$$\text{Product: } z = |z_1| (\cos \theta + i \sin \theta)$$

$$w = |w| (\cos \beta + i \sin \beta)$$

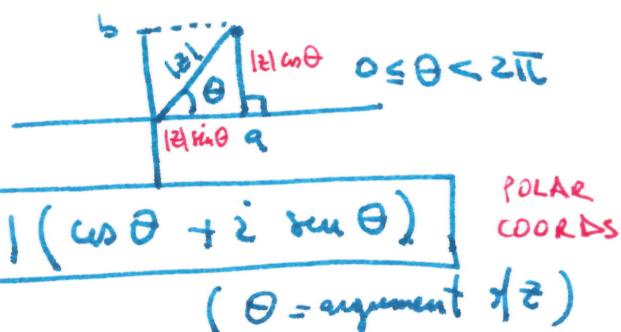
$$z \cdot w = |z| |w| \underbrace{\left( (\cos \theta \cos \beta - \sin \theta \sin \beta) + i (\sin \theta \cos \beta + \cos \theta \sin \beta) \right)}_{= |z \cdot w| \cos(\theta + \beta)} + i \underbrace{\sin(\theta + \beta)}$$

- Multiply modules
- Add the arguments!

Similarly, can take  $n^{\text{th}}$  roots!

$$\text{Find } z \text{ with } z^n = w \quad z = |w|^{\frac{1}{n}} \left( \cos \left( \frac{\text{Arg } w}{n} \right) + i \sin \left( \frac{\text{Arg } w}{n} \right) \right)$$

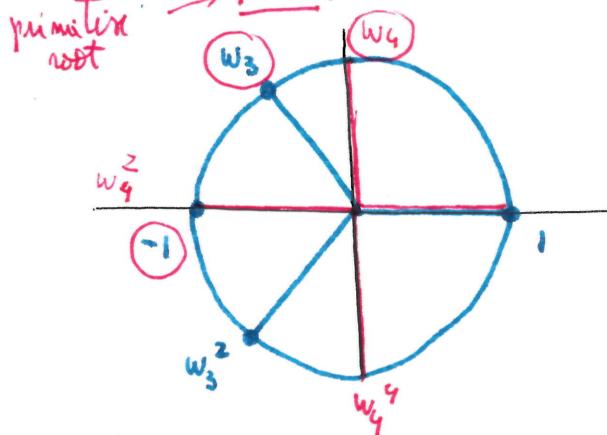
Any other solution:  $\tilde{z}^n = w$  has  $\left(\frac{z}{\tilde{z}}\right)^n = 1$  so it will satisfy  $\tilde{z} = z \tilde{z}^n$  where  $\tilde{z}^n = 1$  (not of unit!)



Why?  $\zeta = (\cos \varphi + i \sin \varphi)$  &  $|z^n| = |\zeta|^n \Rightarrow |\zeta| = 1$ . ) 44

$$\zeta^n = \cos n\varphi + i \sin n\varphi = 1 + i \cdot 0 \quad \text{so } n\varphi \text{ is } 0, \pm 2\pi, \pm 4\pi, \dots$$

so  $\varphi = \frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \dots, \left(\frac{n-1}{n}\right)2\pi \quad \text{exactly } \frac{n}{n} \text{ of them!}$



$$n=2 : \varphi = 0 \& \pi$$

$$n=3 : \varphi : 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$n=4 : \varphi : 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$$

$$\bullet x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \dots + x + 1)$$

EULER'S FORMULA

$$\bullet \text{Def: } e^{i\theta} = \cos \theta + i \sin \theta$$

all roots  $\zeta$  except  $\zeta=1$  make this poly vanish.

$$\Rightarrow e^{i\pi} = \cos \pi + i \sin \pi = -1$$

$$\text{Why? } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{ROC} = \infty \quad x \in \mathbb{R}$$

$$\text{Reflon } x=i\theta \quad e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i)^n \theta^n}{n!} = \underbrace{1}_{\text{Even powers}} + i \underbrace{\theta}_{\text{Odd powers}} - \frac{\theta^2}{2!} + i \frac{\theta^3}{3!} - \frac{\theta^4}{4!} + i \frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i \frac{\theta^7}{7!} \dots$$

Even powers have no  $i$ , odd powers have  $i$ :

group them together (OK b/c ROC =  $\infty$ )

$$= \left( 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \right) + i \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \right)$$

$$= \cos(\theta) + i \sin(\theta)$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$\Rightarrow e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta.$$

$$\Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\& \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$