Exercise 1. **[6 points]** Find the equations for the two lines through the point $(1, 13)$ that are tangent to the parabola $y = 6x - x^2$.

Exercise 2. **[12 points]** Compute the following limits or show that they do not exist:

(i) $\displaystyle \lim_{x\to\infty} \frac{x\tan(2/x)}{\sqrt{1+x}}$, (ii) $\displaystyle \lim_{x\to 1} \left(\frac{1}{x-1} + \frac{1}{x^2-3x+2}\right)$.

Exercise 3. **[8 points]** Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \geq 0, \\ a\,x + b & \text{if } x < 0, \end{cases}$$

where a and b are constants. Find all values of a and b satisfying:

(i) $f(x)$ is a continuous function,

(ii) $f(x)$ is a differentiable function.

Exercise 4. **[12 points]** Consider the circle defined by the equation $x^2 + y^2 = R^2$.

(i) An isosceles triangle is inscribed in the circle with its base parallel to the x-axis and one vertex at the point $(0, R)$. Find the height of the triangle with maximum area and show that this triangle is equilateral with sides of length $\sqrt{3}R$.

(ii) Assume the radius of the circle grows with time at a constant rate of 5 in./s, with an initial radius of 1 in. Find the rate of change of the triangle with maximum area when the radius is 2 in.

Exercise 5. **[12 points]** Consider the function $f(x) = x^3 + 3\,x^2 + 9\,|x| + 7$.

(i) Does the function f have an absolute maximum and minimum value on the interval $[-4, 1]$? If so, find them.

(ii) Does the function f have any inflection points on the interval $[-4, 1]$? if so, find them.

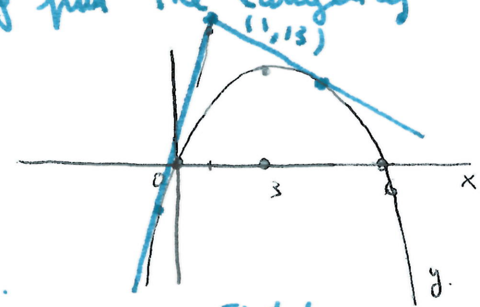
(iii) Find the region in $[-4, 1]$ where f is increasing or decreasing and where it is concave up / down.

(iv) Use the information from the previous items to sketch the graph of the function on the interval $[-4, 1]$.

Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5	TOTAL

SOLUTIONS MIDTERM I

<u>Problem 1</u>: To determine the lines, we need a point $(x, 6x-x^2)$ on the parabola & the slope has to agree with the one coming from the tangency condition.



Sketch

. Tangency gives $\quad y' = 6-2x$

. Point condition gives $\quad \text{slope} = \dfrac{6x-x^2-13}{x-1}$.

So x must satisfy: $\quad 6-2x = \dfrac{6x-x^2-13}{x-1}$

We solve for x and expect 2 solutions:

$$(6-2x)(x-1) = 6x-x^2-13$$

$$6x-6-2x^2+2x = 6x-x^2-13 \rightsquigarrow \quad x^2-2x-7=0$$

$$x = \frac{2\pm\sqrt{4+4\cdot 7}}{2}$$

$$x = \frac{2\pm\sqrt{32}}{2} = \frac{4\pm 2\sqrt{2}}{2} = \boxed{1\pm 2\sqrt{2}}$$

So the points are $\left(1+2\sqrt{2},\ 6(1+2\sqrt{2})-(1+2\sqrt{2})^2\right)$

$\left(1-2\sqrt{2},\ 6(1-2\sqrt{2})-(1-2\sqrt{2})^2\right)$

slopes: $6-2(1+2\sqrt{2}) = 4-4\sqrt{2}$ & $6-2(1-2\sqrt{2}) = 4+4\sqrt{2}$

<u>Equations</u>: Line 1: $y = (4-4\sqrt{2})(x-1)+13$

Line 2: $y = (4+4\sqrt{2})(x-1)+13$.

Problem 2: (i) We use $\tan\left(\frac{2}{x}\right) = \dfrac{\operatorname{sen}\left(\frac{2}{x}\right)}{\cos\frac{2}{x}}$ & $h = \frac{1}{x} \xrightarrow[x\to\infty]{} 0^+$

so lim becomes $\displaystyle\lim_{h\to0^+} \frac{\frac{1}{h}}{\sqrt{1+\frac{1}{h}}} \frac{\operatorname{sen} 2h}{\cos 2h} = \lim_{h\to0^+} \frac{\operatorname{sen} 2h}{2h} \frac{2}{\cos 2h} \frac{1}{\sqrt{1+\frac{1}{h}}} = \boxed{0}$

$\frac{\operatorname{sen} 2h}{2h} \xrightarrow{h\to0^+} 1$

$\sqrt{1+\frac{1}{h}} \xrightarrow{h\to0^+} +\infty$

$\frac{2}{\cos 2h} \to \frac{2}{1} = 2$

(ii) The limit is of the form $\frac{1}{0} + \frac{1}{0}$ so we factor x^2-3x+2

$x^2-3x+2 = (x-1)(x-2)$

$$\lim_{x\to1} \frac{1}{x-1} + \frac{1}{(x-1)(x-2)} = \lim_{x\to1} \frac{1}{x-1}\left(1+\frac{1}{x-2}\right)$$

$$= \lim_{x\to1} \frac{1}{x-1}\,\frac{x-2+1}{x-2} = \lim_{x\to1} \frac{1}{\cancel{x-1}}\,\frac{\cancel{x-1}}{x-2} = \boxed{-1}$$

$\frac{1}{x-2} \xrightarrow{x\to1} \frac{1}{1-2} = \frac{1}{-1} = -1$

Problem 3: (i) For the function to be continuous we need

$\lim_{x\to c} f(x) = f(c)$ for every c. When $c\neq0$, this is true because on each region f is a product, sum or composition of continuous (and differentiable) functions.

We compute the side limits $\displaystyle\lim_{x\to0^-} f(x) = \lim_{x\to0^-} ax+b = b$

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x^2 \underbrace{\operatorname{sen}\frac{1}{x}}_{\text{bounded}} = \lim_{x\to0^+} 0\cdot\text{bounded} = 0$$

$\boxed{\text{So for continuity we MUST have } b=0 \text{ \& } a \text{ is free}}$

(ii) For differentiable, we already know $b=0$ (because differentiable functions are continuous. So $f(0)=0$.

On each piece, f is differentiable, so we need to check $f'(0)$.

We use side limits.

- $$\lim_{\Delta x \to 0^+} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{(\Delta x)^2 \sin \frac{1}{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0^+} (\Delta x) \sin \frac{1}{\Delta x} = \boxed{0}$$

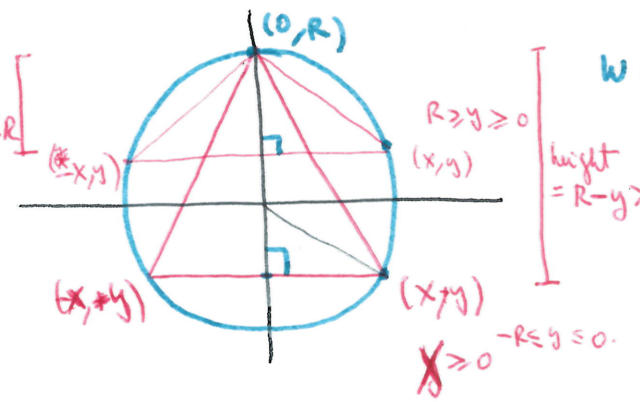
(Again, we use "0 . bounded $=0$")

- $$\lim_{\Delta x \to 0^-} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{a(\Delta x) - 0}{\Delta x} = a.$$

$\boxed{\text{So for } f \text{ to be differentiable we MUST have } a=0=b.}$

Problem 4 :

height $= R-y<R$



We draw the triangle & use the symmetry of the picture to conclude that the 2 missing vertices have coordinates $(x,+y)$ & $(-x,y)$

$$\text{Area} = \frac{\text{base} \cdot \text{height}}{2} = \frac{2x \cdot \text{height}}{2} = x \text{ height}.$$

- By looking at the picture we know that the maximum area must occur for $y<0$. ~~If that~~ (If not, the triangle with vertices $(-x,-y)$, $(+x,-y)$ will have the same length for the base & its height will be $R+y > R-y$)

height $= R-y$. $(-R \leq y \leq 0)$ $\rightsquigarrow$ Area $= x(R-y)$.

x & y are related by $x^2+y^2=R^2$

Maximizing the area is the same as maximizing the square of the area

So we want to maximize the function $F_{(y)} = x^2(R-y)^2$
$= (R^2-y^2)(R-y)^2$
on the region $[-R, 0]$. (differentiable!)

To solve, we find the critical values & compare with $F_{(0)} = F_{(-R)} = 0$

$$F' = (-2y)(R-y)^2 + (R^2-y^2)(2(R-y)(-1))$$
$$= (R-y)^2(-2y - 2(R+y))$$
$$= (R-y)^2(-2)(2y+R) = 0 \quad \text{gives } \boxed{y = -\frac{R}{2}}$$

So the height $= R + \frac{R}{2} = \boxed{\frac{3R}{2}}$ & $F_{(-\frac{R}{2})} > 0$ so it's maximal.

This gives $x = \sqrt{R^2 - \left(\frac{R}{2}\right)^2} = R\sqrt{1-\frac{1}{4}} = \frac{R\sqrt{3}}{2}$.

The length of the base is $\boxed{R\sqrt{3}}$.

The length of the other 2 sides are $\sqrt{\left(\frac{R\sqrt{3}}{2}\right)^2 + \left(\frac{3R}{2}\right)^2} = \sqrt{R^2 3} = \sqrt{3}R$

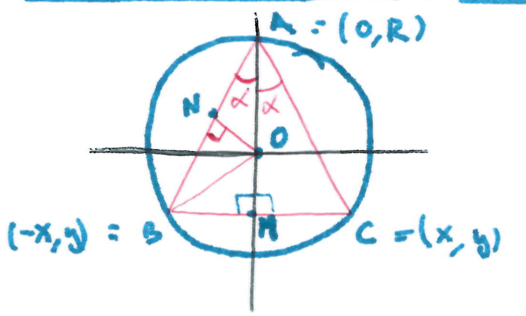
so we get that the triangle with maximal area is equilateral & the area is given by $A(R) = \frac{R\sqrt{3}}{2}\frac{3R}{2} = \boxed{\frac{3\sqrt{3}}{4}R^2}$

(ii) We write $R = R(t)$.

The rate of change of $A(R_{(t)})$ is

$$\frac{d}{dt}A(R_{(t)}) = \frac{3\sqrt{3}}{4}2R \cdot R'_{(t)} = \frac{3\sqrt{3}}{4}2\cdot 2\cdot 5 = \boxed{\sqrt{3}\cdot 15 \ in^2/s}$$

Alternative Solution for (i): Use the fact that equilateral triangles have angles of value $60° = \frac{\pi}{3}$.



As before, two vertices must be below the x-axis. We label the vertices of the Δ by A, B, C, the origin & draw a right triangle with a vertex along one of the diagonal sides of the triangle. We call the vertex on this side by N

We let M be the midpoint of BC

By construction, the angle $\hat{BAM} = \hat{MAC}$. We call it α.

Since $\overline{OB} = \overline{OA} = R$, the segment ON is the height of the triangle $\hat{BMA}$

In particular: $AM = AB \cos\alpha$ But $AB = 2AN = 2\cos\alpha \underset{AO}{\underset{\shortparallel}{R}}$

$BM = AB \sin\alpha$

So we get $AM = 2R\cos^2\alpha$ & $BM = 2R\cos\alpha \operatorname{sen}\alpha = R\sin(2\alpha)$

Then: $\text{Area } \underset{(\alpha)}{\hat{ABC}} = \frac{BC \cdot AM}{2} = BM \cdot AM = 4R^2\cos^3\alpha \operatorname{sen}\alpha \quad 0 \leq \alpha \leq \frac{\pi}{2}$

For maximal area, we check for crit pts since Area continuous (0) = Area$\left(\frac{\pi}{2}\right) = 0$.

$$A'(\alpha) = 4R^2\left(3\cos^2\alpha\,(-\operatorname{sen}\alpha)\operatorname{sen}\alpha + \cos^3\alpha\cos\alpha\right)$$
$$= 4R^2\cos^2\alpha\left(-3\operatorname{sen}^2\alpha + \cos^2\alpha\right)$$

So $A'(\alpha) = 0$ forces either $\cos\alpha = 0$ or $\cos^2\alpha = 3\operatorname{sen}^2\alpha$ (for $0 < \alpha < \frac{\pi}{2}$)

Since $\cos\alpha \neq 0$ on $(0, \frac{\pi}{2})$, we must have $\cos^2\alpha = 3\operatorname{sen}^2\alpha$

Since $\cos^2\alpha = 1 - \operatorname{sen}^2\alpha$ we get $1 - \operatorname{sen}^2\alpha = 3\operatorname{sen}^2\alpha$ so $\operatorname{sen}^2\alpha = \frac{1}{4}$

This gives $\sin\alpha = \pm 1/2$ for $0 \leq \alpha \leq \frac{\pi}{2}$

The only solution is $\alpha = \frac{\pi}{6}$ Then $\hat{CAB} = 2\alpha = \frac{\pi}{3}$

Since the other 2 angles are equal, their value is $\frac{1}{2}\left(\pi - \frac{\pi}{3}\right) = \frac{\pi}{3}$.

So all angles are equal & we get that the Δ is equilateral.

Value of Area $= 4R^2\cos^3\frac{\pi}{6}\operatorname{sen}\frac{\pi}{6} = \frac{3\sqrt{3}}{4}R^2$ (side length $= 2R\cos\frac{\pi}{6} = \sqrt{3}R$)

Problem 5 : (i) The function is not differentiable at $x=0$.

So crit pts are $\underline{x=0}$ & those where $f'(x)=0$.

For $x>0$ $\quad f'(x) = 3x^2+6x+9 = 3(x^2+2x+3) = 3((x+1)^2+2)$

For $x<0$ $\quad f'(x) = 3x^2+6x-9 = 3(x^2+2x-3) = 3((x+1)^2-4)$

$$= 3((x+3)(x-1)).$$

So for $x>0$ $f'(x)>0$ always.

So for $x<0$ $f'(x)=0$ if and only if $x=-3>-4$.

$f''(x) = 6x+6 \qquad f''(x) < 0$ for $x<-1$ so $x=-3$ is a MAX

We know f has a max & a min because f is continuous so EVT applies to $[-4,1]$.

$f(0)=7$, $\quad f(1) = 1+3+9+7 = 20$, $\quad f(-4) = -64+48+36+7 = 27$

$f(-3) = -27+27+27+7 = 34$

$$\boxed{\text{MAX at } x=-3 \ \ \& \ \ \text{MIN at } x=0}$$

(ii) We compute $f''(x) = 6x+6$ for $x \neq 0$ so only change in convexity is at $x=-1$. we have an inflection pt there.

(iii)

	-4 to -3	-3 to -1	-1 to 0	0 to 1
sgn f''	$-$	$-$ 0	+ +	+
sign f'	+ 0	$-$	$-$	+
f	CD ↗	CD ↘	CU ↘	CU ↗

$f(-1) = -1+3+9+7 = 18$

(iv)