Exercise 1. **[8 points]** A ball is thrown vertically upward with from the roof of a building 400 ft high an initial velocity of 78 ft/s. Find the distance s from the ground up to the ball t seconds later. If the ball misses the building on the way down, how long does it take for it to hit the ground?

Exercise 2. **[10 points]** Compute the following limit and derivative (if possible):

$$\text{(i)}\ \lim_{x\to 0}\frac{\sin(2x)}{e^x-1}, \qquad \text{(ii)}\ \frac{d}{dx}\int_0^{\cos(x)}\frac{t\,dt}{\sqrt{1+t^2}}.$$

Exercise 3. **[8 points]** Consider a spring of length 3 ft. Show that the work done in stretching it from a length of 4 ft. to a length of 5 ft. is one-fourth of the work done in stretching it from 5 ft. to 7 ft.

Exercise 4. **[12 points]** Consider the region bounded by the curves $y = x^2 - 2$, $y = 1$, $x = 1$, and $x = -1$

(i) Sketch the region and compute its area.

(ii) Find the volume of the solid obtained by rotating the region about the $y = 1$ line.

Exercise 5. **[12 points]** Show that the surface of revolution generated by the loop of equation $18y^2 = x(6-x)^2$ when revolved about the x-axis has surface area 12π.

Bonus Problem: Show that the equation $x = \ln(x)$ has no solution in two ways:

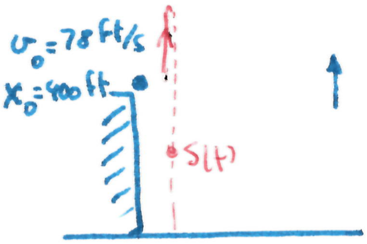
(i) by minimizing the function $y = x - \ln(x)$,

(ii) geometrically, by considering the graph of the functions $y = x$ and $y = \ln(x)$.

Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5	TOTAL	Bonus

Solutions - Midterm 2

Problem 1:



$\uparrow F = -mg$
$= -m\, 32\, ft/s^2$

$m a(t) = -mg$ so $a(t) = -32$

Integrate to get velocity

$$v(t) = \int_0^t a(x)\,dx = -32t + C$$

By initial conditions: $v(0) = v_0 = 78 = 0 + C$ so $C = 78$

$v(t) = -32t + 78$. Integrate again to get $s(t)$ = trajectory

$$s(t) = \int_0^t (-32x + 78)\,dx = -32\frac{t^2}{2} + 78t + C$$

By initial conditions: $400 = s(0) = C$ so $C = 400$

Conclusion: $\boxed{s(t) = -16t^2 + 78t + 400}$

- The ball hits the ground when $s(t) = 0$ so we need to solve:

$$-16t^2 + 78t + 400 = 2(-8t^2 + 39t + 200) = 0$$

$$t = \frac{-39 \pm \sqrt{39^2 + 4 \cdot 8 \cdot 200}}{2(-8)} = \frac{39 \pm \sqrt{(39)^2 + 6400}}{16} = \frac{39 \pm \sqrt{7921}}{16}$$

$$= \frac{39 \pm 89}{16}$$

$\nearrow$ $128/16 = 8$

$\searrow$ $\frac{-50}{16} < 0 \leadsto$ discard.

Answer: In 8 seconds it hits the ground.

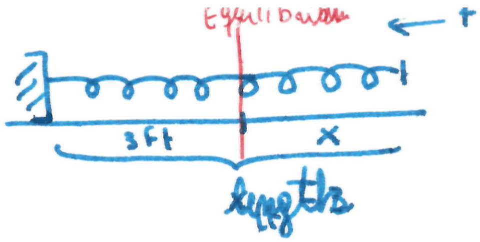
Problem 2

(i) $$\lim_{x\to 0} \frac{\sin 2x}{e^x - 1} = \lim_{x\to 0} \boxed{\frac{\sin 2x}{2x}}^{\xrightarrow{x\to 0} 1} \cdot 2\,\frac{x}{e^x - 1} = 2 \lim_{x\to 0}\left(\frac{e^x - 1}{x}\right)^{-1}$$

$$= 2\left(\lim_{x\to 0}\frac{e^x - 1}{x}\right)^{-1} \underset{\text{def of derivative of } e^x \text{ at } x=0}{=} 2\left((e^x)'\big|_{x=0}\right)^{-1} = 2\cdot\left(e^x\big|_{x=0}\right)^{-1} = 2 \cdot 1^{-1} = \boxed{2}$$

(ii) Use FTC $\quad \underline{A} = \frac{\cos x}{\sqrt{1 + \cos^2 x}} (\cos x)' = \boxed{\frac{-\cos x \sin x}{\sqrt{1 + \cos^2 x}}}$

Problem 3



We know $F_{(x)} = -kx$
restorative force

$x = \text{length} - 3$

$$W = \int_{x_{in}}^{x_{term}} F_{(x)}\,dx = \int_{x_{in}}^{x_{term}} -kx\,dt = -k\frac{x^2}{2}\Big|_{x_{in}}^{x_{term}}$$

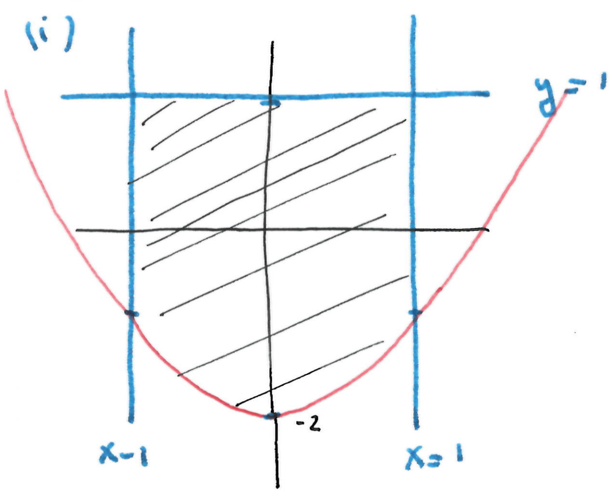
So $W_{4\to5} = -k\left(\frac{x^2}{2}\Big|_{4-3=1}^{5-3=2}\right) = -\frac{k}{2}(2^2-1^2) = -k\frac{3}{2}$

$W_{5\to7} = -\frac{k}{2}\,x^2\Big|_{5-3=2}^{7-3=4} = -\frac{k}{2}(16-2^2) = -\frac{k}{2}12$

Conclude: $W_{5\to7}/W_{4\to5} = \dfrac{-\frac{k}{2}12}{-\frac{k}{2}3} = \boxed{4}$.

Problem 4

(i)

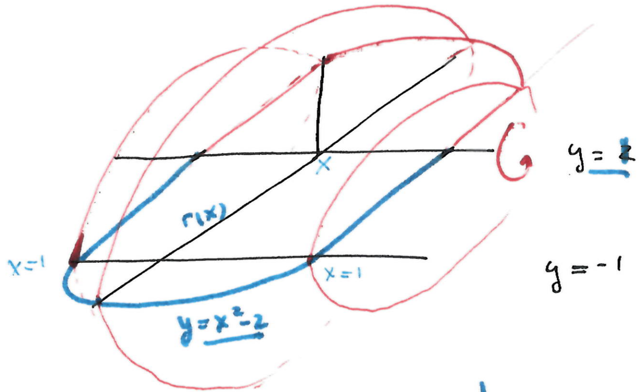


Compute intersections

- $x^2-2 \stackrel{?}{=} 1$
  $x^2 = 3$
  $x = \pm\sqrt{3}$ outside $|x|\leq 1$.
- $x^2-2 = y$ & $x=\pm 1$ gives
  $y = (\pm1)^2-2 = -1$.

$$\text{Area} = \int_{-1}^{1}(1-(x^2-2))\,dx = \int_{-1}^{1}(3-x^2)\,dx = 3x-\frac{x^3}{3}\Big|_{-1}^{1}$$

$$= 3-\frac{1}{3}-\left(-3+\frac{1}{3}\right) = 2\left(\frac{8}{3}\right) = \boxed{\frac{16}{3}}$$



Slice along x-axis

$A_{(x)}$ = circle of radius $r_{(x)} = 1-(x^2-2)$
$= 3-x^2$

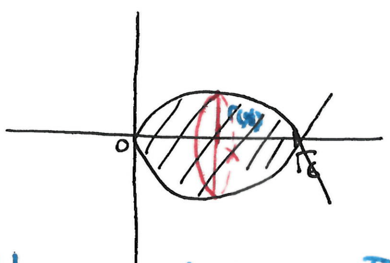
Limits = $x=1$ & $x=-1$

$$\text{Vol} = \int_{-1}^{1} A_{(x)}\,dx = \int_{-1}^{1} \pi(3-x^2)^2\,dx = \int_{-1}^{1} \pi(9+x^4-6x^2)\,dx$$

$$= \pi\left(9x+\frac{x^5}{5}-2x^3\Big|_{-1}^{1}\right) = \pi\left(\left(9+\frac{1}{5}-2\right)-\left(-9-\frac{1}{5}+2\right)\right)$$

$$= 2\pi\left(7+\frac{1}{5}\right) = 2\pi\,\frac{36}{5} = \boxed{\frac{72\pi}{5}}$$

Problem 5 Equation is $\underbrace{18y^2}_{\geq 0} = x\underbrace{(6-x)^2}_{\geq 0}$ so x must be ≥ 0.



$$A = \int_0^6 2\pi\, r_{(x)}\,ds = \int_0^6 2\pi\, y_{(x)}\sqrt{1+y'^2_{(x)}}\,dx$$

Use implicit differentiation to get y'.

$$2\cdot 18\, y\, y' = (6-x)^2 + 2x(6-x)(-1) = (6-x)(-2x+6-x)$$
$$= (6-x)(-3x+6)$$

$$y' = \frac{(6-x)(6-3x)}{36y}$$

Now, replace in $\sqrt{1+y'^2} = \sqrt{1+\frac{(6-x)^2(6-3x)^2}{36^2y^2}} = \frac{1}{36y}\sqrt{36^2y^2+(6-x)^2(6-3x)^2}$

$$= \frac{1}{36y}\sqrt{36^2\,\frac{1}{18}x(6-x)^2+(6-x)^2(6-3x)^2} = \frac{6-x}{36y}\sqrt{4\cdot 18x+(6-3x)^2}$$

use $y^2 = \frac{1}{18}x(6-x)^2$

$$\underbrace{4\cdot 18}_{4\cdot 8}x + \underbrace{(6-3x)^2}_{9(2-x)^2} = 108x+36+9x^2-36x^2 = 9(8x+4+x^2-4x)$$
$$= 9(x^2+4x+2) = 9(x+2)^2$$

We get $\sqrt{1+y'^2} = \frac{6-x}{36y}\sqrt{9(x+2)^2} = \frac{(6-x)(x+2)}{4y}$

So $A = \int_0^{6} 2\pi\, y \,\frac{(6-x)(x+2)}{4y}\,dy = \frac{\pi}{2}\int_0^6 (6x+12-x^2-2x)\,dx$

$= \frac{\pi}{2}\int_0^6 12-x^2+4x\,dx = \frac{\pi}{2}\left(12x-\frac{x^3}{3}+2x^2\Big|_0^6\right)$

$= \frac{\pi}{2}\left(12\cdot 6 - 2\cdot 36 + 2\cdot 36\right) = \boxed{36\pi}$

BONUS : (1) $y = x - \ln(x)$ defined on $x > 0$, cont & diff'ble

So it's minimum (if it exists), will have $y' = 0$.

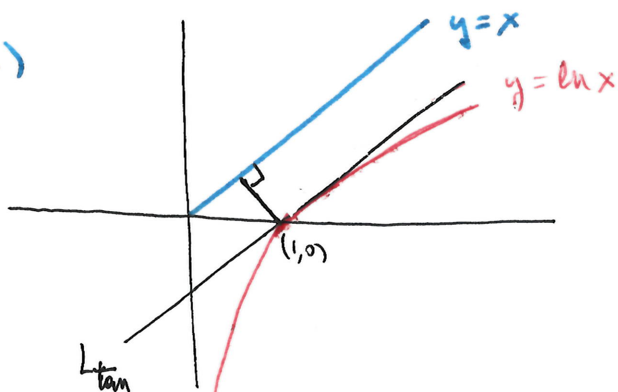
But $y' = 1 - \frac{1}{x} = \frac{x-1}{x} = 0$ gives $\boxed{x=1}$

Note $y' < 0$ for $x < 1$ ⇝ decreasing.
$y' > 0$ for $x > 1$ ⇝ increasing. } $x=1$ is indeed the minimum

Min value is $y(1) = 1 - \ln(1) = 1 - 0 = 1 > 0$.

So $\boxed{x - \ln x \geq 1 > 0 \text{ for all } x}$ & so $x = \ln x$ has no solution.

(2)



Check the convexity of $\ln x$.

$y' = \frac{1}{x}$, $y'' = \frac{-1}{x^2} < 0$

so it's convex down

So tangent line L_{tan} at $x=1$

$y = \frac{1}{1}(x-1) = x - 1$

So slope is 1 & curve lies below it. Since L_{tan} is below the line $x=y$, this ensures $x > \ln x$ always & so no solution to $x = \ln x$ can exist.