Exercise 1. **[8 points]** Compute $\displaystyle\lim_{x\to\pi} \frac{\sin^2(x)}{1+\cos(3\,x)}$.

Exercise 2. **[12 points]** Decide if the series below are absolutely convergent, conditionally convergent, or divergent. Be sure to state which test you are using and verify the conditions.

(i) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+3}$ (ii) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin^2(n)}{n^{n/2}}$

Exercise 3. **[10 points]** Compute $\displaystyle\int_0^{\infty}(x+2)\,e^{-x}d\,x.$

Exercise 4. **[10 points]** Compute $\displaystyle\int_1^{\sqrt{3}}\frac{(3\,x^3+5\,x^2+x+4)\,d\,x}{(x^4+x^2)}.$

Exercise 5. **[10 points]** Compute $\displaystyle F(t) := \int_0^t \frac{1-\sin(2x)}{\cos^2(2x)}dx$ for $0\le t<\pi/4$.

Bonus problem: Decide if the function F from Exercise 5 can be extended continuously to $t=\pi/4$, and if so, find $F(\pi/4)$.

Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5	TOTAL	Bonus

Solutions - Midterm 3

Problem 1: $\lim_{x\to\pi} \frac{\sin^2(x)}{1+\cos(3x)}$ has the form $\frac{0}{1-1}=\frac{0}{0}$ so we use L'Hospital

$$\lim_{x\to\pi} \frac{\sin^2 x}{1+\cos 3x} = \lim_{x\to\pi} \frac{2\sin x\cos x}{-3\sin 3x} = -\frac{2}{3}\lim_{x\to\pi} \frac{\overset{\to 0}{\sin x}\ \overset{\to -1}{\cos x}}{\underset{\to 0}{\sin 3x}}$$

$$= \frac{2}{3}\lim_{x\to\pi}\underbrace{\frac{\sin x}{\sin 3x}}_{\frac{0}{0}} \underset{\text{L'Hop again}}{=} \frac{2}{3}\lim_{x\to\pi}\frac{\cos x}{3\cos 3x} = \frac{2}{9}\lim_{x\to\pi}\frac{\cos x}{\cos 3x}$$

$$= \frac{2}{9}\frac{(-1)}{(-1)} = \boxed{\frac{2}{9}}$$

Problem 2: We start by checking for absolute convergence.

(1) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+3}$ Use comparison with $\frac{1}{\sqrt{n}}=\frac{1}{n^{1/2}}$ p-series with $p=\frac{1}{2}<1$ so divergent

- $\frac{\sqrt{n}}{n+3} \Big/ \frac{1}{\sqrt{n}} = \frac{n}{n+3} \xrightarrow[n\to\infty]{} 1 \neq 0$ Limit comparison says $\sum \frac{\sqrt{n}}{n+3}$ diverges

- We need to check convergence of $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sqrt{n}}{n+3}$. Use Alternating Series Test

  - $a_n = \frac{\sqrt{n}}{n+3} \xrightarrow[n\to\infty]{} 0$ ✓

  - $a_{n+1} \le a_n$ for n large enough? $\frac{\sqrt{n+1}}{(n+1)+3} \overset{?}{\le} \frac{\sqrt{n}}{n+3}$

$$\frac{\sqrt{n+1}}{\sqrt{n}}\ \frac{n+3}{n+4} \overset{?}{\le} 1.$$

This is true because $\sqrt{\frac{n+1}{n}}\ \frac{n+3}{n+4} = \sqrt{\frac{(n+1)\ (n+3)^2}{n\ \ (n+4)^2}} \overset{?}{\le} 1.$

This is true if and only if $\frac{(n+1)(n+3)^2}{n\ (n+4)^2} \le 1.$ $\boxed{\text{for } n \text{ large enough}}$

$$(n+1)\ (n+3)^2 \le n\ (n+4)^2 = n\left(n^2+8n+16\right)$$

$$= (n+1)\left(n^2+6n+9\right) = n^3+7n^2+15n+9$$

Enough to check: $n^3+7n^2+15n+9 \leq n^3+8n^2+16n$ for n large enough

$$7n^2+15n+9 \leq 8n^2+16n$$

$$0 \leq n^2+n-9$$

We factor $n^2+n-9=0 \rightsquigarrow \dfrac{-1\pm\sqrt{1+4\cdot 9}}{2} = \dfrac{-1\pm\sqrt{37}}{2}$

So $(n^2+n-9) = \left(n+\dfrac{1-\sqrt{37}}{2}\right)\left(n+\dfrac{1+\sqrt{37}}{2}\right) \geq 0$ for $n \geq \dfrac{\sqrt{37}-1}{2}$

true for $n \geq \dfrac{7-1}{2} = 3$ ✓

Conclusion: $a_{n+1} \leq a_n$ if $n \geq 3$.

AST shows $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sqrt{n}}{n+3}$ $\boxed{\text{converges conditionally}}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{5/2}}$ converges because $\left|\dfrac{\sin^2 n}{n^{5/2}}\right| \leq \dfrac{1}{n^{5/2}} \leq \dfrac{1}{n^2}$ if $n \geq 4$.

By comparison with p-series for $p=2$ we have $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^{5/2}}$ converges absolutely

So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin^2 n}{n^{5/2}}$ $\boxed{\text{converges absolutely}}$.

Problem 3: $\displaystyle\int_0^{\infty} (x+2)\,e^{-x}\,dx$ is an improper integral. $\boxed{\text{ANSWER} = 3}$

$$\int_0^t (x+2)\,e^{-x}\,dx = \int_0^t xe^{-x}\,dx + \int_0^t 2e^{-x}\,dx$$

int by Parts

$$\int_0^t \underbrace{x}_{v}\underbrace{e^{-x}dx}_{du} = -xe^{-x}\Big|_0^t - \int_0^t -e^{-x}\,dx = -xe^{-x}\Big|_0^t - e^{-x}\Big|_0^t$$

$u = -e^{-x}$

$$\text{So } \int_0^t (x+2)e^{-x}\,dx = \left(-xe^{-x}-e^{-x}-2e^{-x}\right)\Big|_0^t = \left(-xe^{-x}-3e^{-x}\right)\Big|_0^t$$

$$= \left(-te^{-t}-3e^{-t}\right) - \left(-0-3\right) = 3 - \frac{t}{e^t} - \frac{3}{e^t}$$

$$\lim_{t\to 0}\int_0^t (x+2)\,e^{-x}\,dx = \lim_{t\to\infty} 3 - \frac{t}{e^t} - \frac{3}{e^t} = 3 - \lim_{t\to\infty}\frac{t}{e^t} \underset{\text{L'Hôpital}}{=} 3 - \lim_{t\to\infty}\frac{1}{e^t} \to \boxed{3}$$

Problem 4 We use Partial Fractions

denominator $= x^4 + x^2 = x^2(x^2+1)$.

So $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} = \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2}{x^2(x^2+1)}$

We get $3x^3 + 5x^2 + x + 4 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$

$= (A+C)x^3 + (B+D)x^2 + Ax + B$

Equate coeff-by-coeff to get

$3 = A + C \rightsquigarrow C = 3-1 = 2$

$5 = B + D \rightsquigarrow D = 5-4 = 1$

$1 = A$

$4 = B$

$$\int_1^{\sqrt{3}} \frac{1}{x} + \frac{4}{x^2} + \frac{2x+1}{x^2+1}\, dx = \ln x - \frac{4}{x} + \ln(x^2+1) \Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{dx}{x^2+1}$$

$$\int_1^{\sqrt{3}} \frac{dx}{x^2+1} \underset{\downarrow}{=} \int_{\arctan 1}^{\arctan\sqrt{3}} du = u \Big|_{\arctan 1}^{\arctan\sqrt{3}} \underset{\uparrow}{=} \sqrt{3} - 1$$

$x = \tan u \quad x^2+1 = \sec^2 u$

$dx = \frac{1}{\cos^2 u}\, du = \sec^2 u$

$\tan\frac{\pi}{3} = \sqrt{3}$

$\tan\frac{\pi}{4} = 1$

So ANSWER for $\int_1^{\sqrt{3}}$ $= \ln\sqrt{3} - \frac{4}{\sqrt{3}} + \ln(10) - (0 - 4 + \ln 2) + \sqrt{3} - 1$

$$= \boxed{\ln\left(\frac{\sqrt{3}\cdot 10}{2}\right) + 3 - \frac{1}{\sqrt{3}}}$$

ANSWER for $\int_0^{\sqrt{3}}$ = divergent (due to $\ln x$ & $-\frac{4}{x}$: we can't evaluate at $x=0$)

Problem 5

$$F(t) = \int_0^t \frac{1-\text{sen}\,2x}{\cos^2 2x}\,dx \underset{\substack{u=2x \\ du = 2\,dx}}{=} \int_0^{2t} \frac{1-\text{sen}\,u}{\cos^2 u}\,\frac{du}{2}$$

$$= \frac{1}{2}\int_0^{2t} \left(\sec^2 u - \frac{\text{sen}\,u}{\cos^2 u}\right) du = \frac{1}{2}\left(\tan u - \frac{1}{\cos u}\right)\Bigg|_0^{2t}$$

$$= \frac{1}{2}\left(\tan 2t - \frac{1}{\cos 2t} - (0-1)\right) = \frac{1}{2}\left(\frac{\text{sen}\,2t - 1}{\cos 2t} + 1\right) \quad \text{for } 0 \le t < \frac{\pi}{4}$$

Bonus Problem: We need to see if $\lim_{x \to \frac{\pi}{4}^-} F(t)$ exists.

Equivalently, does $\lim_{t \to \frac{\pi}{4}^-} \frac{\text{sen}\,2t - 1}{\cos 2t} = \lim_{t \to \frac{\pi}{2}^-} \left(\frac{\text{sen}\,t - 1}{\cos t}\right)$ exist?

Looks like $\frac{0}{0}$, so we use L'Hôpital

$$\lim_{t \to \frac{\pi}{2}^-} \frac{\text{sen}\,t - 1}{\cos t} = \lim_{t \to \frac{\pi}{2}^-} \frac{\cos t}{-\text{sen}\,t} = \boxed{0}.$$

A: We can extend it in a continuous fashion by $F\left(\frac{\pi}{4}\right) = \frac{1}{2}(0+1) = \frac{1}{2}$