Practice Midterm 1 – Solutions

Problem 1: Use implicit differentiation $y = y(x)$.

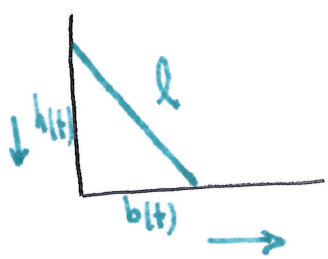
$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1}\cdot y' = 0 \quad \text{so} \quad y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}.$$

This is well defined outside $x=0$, which is true for the point $(-3\sqrt{3}, 1)$.

$$y' = -\sqrt[3]{\frac{1}{-3\sqrt{3}}} = -\sqrt[3]{\frac{1}{-3^{3/2}}} = +\frac{1}{\sqrt{3}}$$

Tangent line: $y = \frac{+1}{\sqrt{3}}(x + 3\sqrt{3}) + 1$

Problem 2:



$$l^2 = b^2(t) + h^2(t) \qquad (*)$$

Constraints: $h'(t) = -0.15\ m/s$

When $b(t) = 3\,m$, $b'(t) = 0.2\ m/s$

Use implicit differentiation:

$$0 = 2b(t)\,b'(t) + 2h(t)\cdot h'(t)$$

For $t = t_0$ we have $b(t_0) = 3$, $b'(t_0) = 0.2$, $h'(t_0) = -0.15$

Evaluate at $t = t_0$:

$$0 = 2\cdot 3\cdot 0.2 + 2\,h(t_0)\,(-0.15)$$

$$0 = 1.2 - 0.3\,h(t_0) \quad \text{so} \quad h(t_0) = 4$$

Replace in the equation $(*)$ to find l:

$$l^2 = b^2(t_0) + h^2(t_0) = 9 + 16 = 25 \quad \text{so} \quad \boxed{l = 5\,m}$$

Problem 3: $f(x) = x^3 - 3x + 2$ is cont & differentiable up to any order.

Crit pts: $f'(x) = 0$. $f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$ ⇝ $x = 1$, $x = -1$ crit pts

discarded!

Crit values: $f(1) = 1 - 3 + 2 = 0$

End points $f(0) = 2$, $f(3) = 27 - 9 + 2 = 20$

By the Extreme Value Thm, we know f has abs. max at $x=3$ & abs. min at $x=1$. There ~~may~~ be other local extrema only among $x=0$ & $x=3$.

We know $x=3$ gives an abs. extremal; so in particular it's local.

To determine if $x=0$ is a local extrema, we compute $f'(0) = -1 < 0$ so the function is decreasing,

Since f' is continuous, $f' < 0$ locally around $x=0$, so at $x=0$ we have a local maximum value.

- To find inflection pts, we compute $f''(x) = 6x$. Only possible inflection point: $x=0$. But there can't be any <u>change</u> in concavity since $x=0$ is an end point.

<u>Conclusion</u>:
- No inflection pts
- $x=0$ local maximum
- $x=1$ local & absolute ~~maximum~~ minimum
- $x=3$ ~~local~~ & absolute maximum.

<u>Problem 4</u>: (1) Use algebra of limits

$$\lim_{x\to\infty} 3x \tan\frac{1}{x} = \lim_{x\to\infty} 3x \frac{\sin\frac{1}{x}}{\cos\frac{1}{x}} \underset{u=\frac{1}{x}}{=} \lim_{u\to 0^+} \frac{3}{u}\frac{\sin u}{\cos u} = \lim_{u\to 0^+} \frac{3}{\cos u}\frac{\sin u}{u}$$

$\frac{\sin u}{u} \to 1$; $\cos 0 = 1$ ($\cos(x)$ is cont)

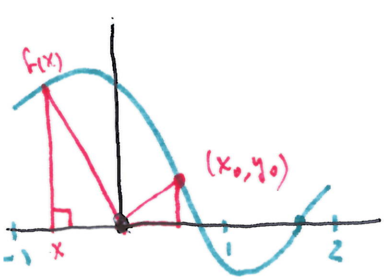
so the limit exists and $= 3$.

(2) $$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} \left(\sqrt{x+1} - \sqrt{x}\right)\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty} \frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = 0.$$

$\sqrt{x+1} \to +\infty$, $\sqrt{x} \to +\infty$

<u>Problem 5</u>: Point closest to the origin = minimizes the distance between $(0,0)$ & $(x, f(x))$



$$\text{Distance} = \sqrt{x^2 + f(x)^2} \geq 0$$

Minimizing the distance is the <u>same</u> as minimizing its square!

$g(x) = x^2 + f(x)^2$ defined on $[-1,2]$. We ~~know~~ know it has a minimum because g is cont & diff'ble!

If the min satisfies $x_0 \neq -1, 2$, then it's not an endpoint & it must be a critical pt! $g'(x) = 2x + 2f(x)f'(x) = 0$

so $f'(x)f(x) = -x$ (1) If the closest point is not at $x_0 = 0$,

then $f'(x_0)\cdot f(x_0) = -x_0 \neq 0$ so $\boxed{f'(x_0) = -\dfrac{x_0}{f(x_0)}}$.

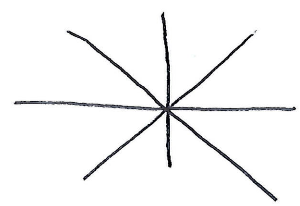
Line through $(0,0)$ & $(x_0, f(x_0))$ is $y = m(x-0)+0$

Slope $= m = \dfrac{f(x_0)-0}{x_0-0} = \dfrac{f(x_0)}{x_0} = \dfrac{-1}{f'(x_0)}$ so it's normal!

(2) If the closest point has $x_0 = 0$, then $f'(x_0)\overset{\neq 0}{f(0)} = 0$ so $f'(0) = 0$ (horizontal!)

But the line through $(0,0)$ & $(0, f(0))$ is $x = 0$ (vertical)

Since a horizontal line is normal to a vertical line, the result also holds for this case.



Problem 6: (a) Note $-1 \le \sin\left(\frac{1}{x}\right) \le 1$

If $x > 0$, we have $-x \le x\sin\frac{1}{x} \le x$

If $x < 0$ we have $u = -x \geq x\sin\frac{1}{x} \geq x = -u$

$\overset{\shortparallel}{-u\sin\left(-\frac{1}{u}\right)} = u\sin\left(\frac{1}{u}\right)$. ⇝ f is even!

We need to find values of x where $x\sin\frac{1}{x} = x$ (that is $\sin\frac{1}{x} = 1$)

& where $x\sin\frac{1}{x} = -x$ (——— $= -1$)

1st case: $\frac{1}{x} = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots$ ⇝ $x = \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \dots$

2nd case: $\frac{1}{x} = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \dots$ ⇝ $x = \frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi}, \dots$

In both cases, we have infinitely many solutions because $\sin(u)$ is periodic.

(b) Away from $x = 0$, $f(x) = x\sin\frac{1}{x}$ it's a product of 2 continuous functions: x & $\sin(\frac{1}{x})$ [this one is a composite of 2 cont. functions]

To show continuity at $x=0$, we use the Squeeze Lemma:

By (a) $\quad -x \le x \sin\frac{1}{x} \le x \quad$ so the middle expression also has $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0 = f(0)$

$$\begin{array}{ccc} -x & & x \\ \downarrow x\to 0 & & \downarrow x \to 0 \\ 0 & & 0 \end{array}$$

(c) ~~None of these limits exist, because we can find points x going to 0 where f(x) = 0 because sin 1/x = 0, e.g. 1/x = kπ so~~

As $x \to +\infty$, we know $u = \frac{1}{x} \to 0^+$ so $\lim\limits_{x\to\infty} x \sin\frac{1}{x}$ can be re-expressed as $\lim\limits_{u\to 0^+} \frac{1}{u}\sin u = 1$

Similarly for $x \to -\infty$. $\quad u = \frac{1}{x} \to 0^-$ so $\lim\limits_{x\to-\infty} x\sin\frac{1}{x} = \lim\limits_{u\to 0^-}\frac{1}{u}\sin u = \boxed{1}$

(d) We can use properties of derivatives = product rule & chain rule, away from $x=0$.

$$f'(x) = \left(x \sin\frac{1}{x}\right)' = \sin\frac{1}{x} + x\cos\frac{1}{x}\left(\frac{-1}{x^2}\right) = \sin\frac{1}{x} - \frac{\cos 1/x}{x}.$$

(e) $$\lim_{x\to\infty} \sin\frac{1}{x} - \frac{\cos 1/x}{x} = \lim_{u\to 0^+} \sin u - u\cos u = 0 - 0 = 0$$

$$\lim_{x\to-\infty} \sin\frac{1}{x} - \frac{\cos 1/x}{x} = \lim_{u\to 0^-} \sin u - u\cos u \overset{\text{sin, prod rule}}{=} 0 - 0 = 0.$$

(f) We use the definition of f':

$$f'(0) = \lim_{\Delta x\to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x \sin\left(\frac{1}{\Delta x}\right)}{\Delta x} = \lim_{\Delta x\to 0} \sin\left(\frac{1}{\Delta x}\right).$$

But this limit does not exist.

- Take $\Delta x = \frac{1}{k\pi}$ for k integer $k \ge 1$, then $\sin\left(\frac{1}{\frac{1}{k\pi}}\right) = \sin(k\pi) = 0$ so the limit should be 0
- Take $\Delta x = \frac{1}{\frac{\pi}{2} + 2k\pi}$ ——— . Then $\sin\left(\frac{1}{\Delta x}\right) = \sin\left(\frac{\pi}{2} + 2k\pi\right) = \sin\left(\frac{\pi}{2}\right) = 1$, so the limit should be 1

We find different values for the limit, so it can't exist!