Solutions Practice Midterm 2

Problem 1:

(1) $\dfrac{\tan 3x}{e^{4x}-1} = \dfrac{\sin 3x}{\cos 3x}\,\dfrac{1}{e^{4x}-1} = \dfrac{\sin 3x}{3x}\,\dfrac{3}{4\cos 3x}\,\dfrac{4x}{e^{4x}-1}$

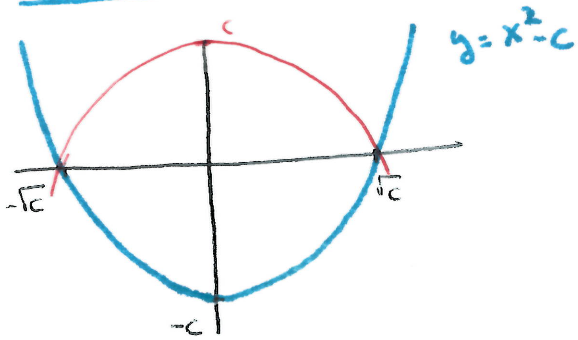
$$\lim_{x\to 0}\frac{4x}{e^{4x}-1} = \lim_{u\to 0}\frac{u}{e^u-1} = \lim_{u\to 0}\frac{1}{\frac{e^u-1}{u}} = \frac{1}{(e^u)'_{(0)}} = \frac{1}{e^0} = 1$$

Conclusion $\displaystyle\lim_{x\to 0}\frac{\tan 3x}{e^{4x}-1} = 1\cdot\frac{3}{4\cdot 1}\cdot 1 = \boxed{\frac{3}{4}}$

(2) Use FTC: $g(u) = \displaystyle\int_0^u \frac{t\,dt}{\sqrt{1+t^2}}$

$$\frac{d}{dx}\left(g(x^5)\right) = \frac{x^5}{\sqrt{1+x^{10}}}\cdot(x^5)' = \frac{x^5}{\sqrt{1+x^{10}}}\cdot 5x^4 = \frac{5x^9}{\sqrt{1+x^{10}}}$$

Problem 2: Start by drawing the curves:



Curve 1: $y = x^2 - c$

Curve 2: $y = c - x^2$

Need to determine their intersection:

$$x^2 - c = c - x^2$$
$$2x^2 = 2c$$
$$\boxed{x = \pm\sqrt{c}}$$

& Curve 2 > Curve 1 ~~at x=0~~

$c > -c$ $(x=0)$

$$\text{Area} = \int_{-\sqrt{c}}^{\sqrt{c}} (\text{Curve 2} - \text{Curve 1})\,dx = \int_{-\sqrt{c}}^{\sqrt{c}} \left(c - x^2 - (x^2 - c)\right)dx$$

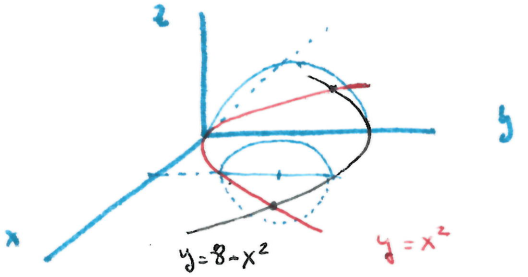
$$= \int_{-\sqrt{c}}^{\sqrt{c}} 2(c - x^2)\,dx = 2\left(cx - \frac{x^3}{3}\right)\Big|_{-\sqrt{c}}^{\sqrt{c}} = 2\left(\left(c\sqrt{c} - \frac{c\sqrt{c}}{3}\right) - \left(-c\sqrt{c} + \frac{c\sqrt{c}}{3}\right)\right)$$

$$= 2\left(\frac{2}{3}c\sqrt{c} - \left(-\frac{2}{3}c\sqrt{c}\right)\right) = \frac{8}{3}c\sqrt{c}$$

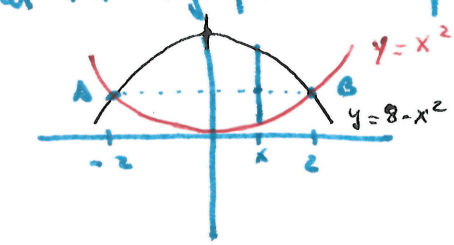
Want Area $=9$ so $9 = \frac{8}{3}c\sqrt{c} = \frac{8}{3}c^{3/2} \rightsquigarrow c^{3/2} = \frac{27}{8}$ so $c = \left(\frac{27}{8}\right)^{2/3}$

$c = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$ We conclude $\boxed{c = \frac{9}{4}}$.

Problem 3 The problem tells us what the cross sections look like.



Look at the xy-plane to find the diameter



Intersection points: $x^2 = 8 - x^2$

$2x^2 = 8$

$x^2 = 4 \rightsquigarrow \boxed{x = \pm 2}$

Height: $8 - x^2 - x^2 = 8 - 2x^2 = 2(4 - x^2) \rightsquigarrow$ radius: $r(x) = 4 - x^2$

$$\text{Volume} = \int_{-2}^{2} \text{Area circle } r(x)\, dx = \int_{-2}^{2} \pi\, r(x)^2\, dx = \int_{-2}^{2} \pi (4 - x^2)^2\, dx$$

↳ cross section is a ~~half~~ full circle of radius $r(x)$

$$\cancel{= \frac{\pi}{2}\left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = \frac{\pi}{2}\left(\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right)}$$

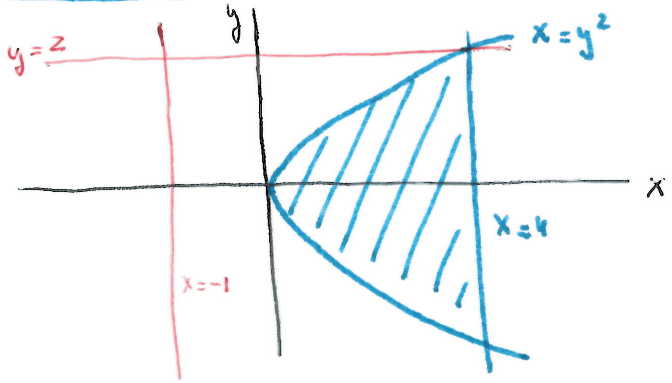
$$\cancel{= \frac{\pi}{2}\left(\frac{2}{3}8 - \left(-\frac{2}{3}8\right)\right) = \frac{\pi}{2}\,\frac{4\cdot 8}{3} = \boxed{\frac{\pi\, 16}{3}}}$$

$$= \pi \int_{-2}^{2} (16 + x^4 - 8x^2)\, dx = 2\pi \int_{0}^{2} (16 + x^4 - 8x^2)\, dx$$

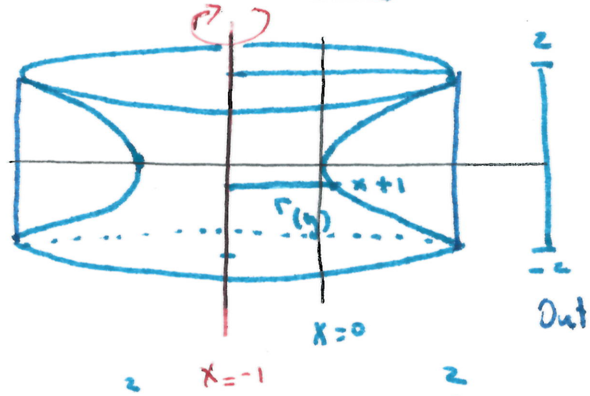
$$= 2\pi\left(16x + \frac{x^5}{5} - \frac{8}{3}x^3\right)\Big|_0^2 = 2\pi\left(32 + \frac{32}{5} - \frac{64}{3}\right)$$

$$= 2\pi\left(\frac{\overset{9\cdot 64}{\overset{\shortparallel}{18\cdot 32}} - 5\cdot 64}{15}\right) = \frac{128\pi\,(4)}{15} = \boxed{\frac{512\pi}{15}}$$

Problem 4: As usual, we start by drawing



(1) Rotate about $x=-1$:



Use y-cross sections & Disk Method for inside

Outside = cylinder of radius $= 1+4 = 5$ & ht $= 4$

. radius $r(y) = 1+x = 1+y^2$

. Limits of integration : $x=4$ & $x=y^2$ give $y=\pm 2$

Outside Vol $= \pi\, 5^2 \cdot 4 = \boxed{100\,\pi}$

$$\text{Inside Vol} = \int_{-2}^{2} A(y)\,dy = \int_{-2}^{2} \pi(1+y^2)^2\,dy \underset{\uparrow}{=} 2\int_0^2 \pi(1+y^4+2y^2)\,dy$$

Symmetry

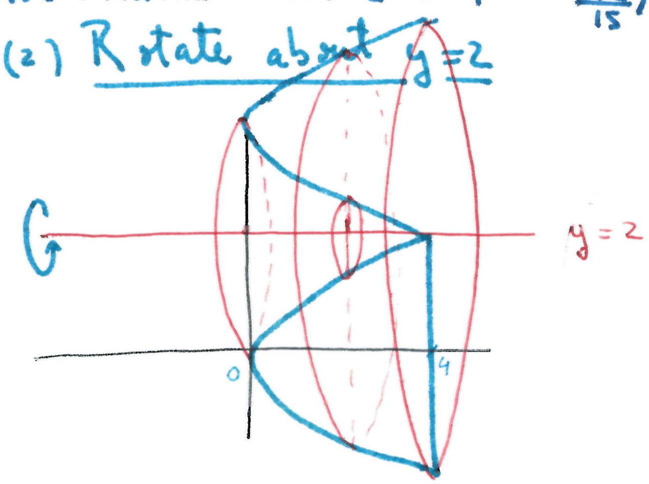
$$= 2\pi\left(y+\frac{y^5}{5}+\frac{2}{3}y^3\right)\Big|_0^2 = 2\pi\left(2+\frac{32}{5}+\frac{16}{3}\right) = 4\pi\left(1+8\left(\frac{2}{5}+\frac{1}{3}\right)\right)$$

$$= 4\pi\left(1+\frac{8\cdot 11}{15}\right) = \frac{4\pi}{15}(15+88) = \boxed{\frac{4\pi\ 103}{15}}$$

$$\text{Vol} = \text{Outside} - \text{Inside} = \pi\left(100-\frac{412}{15}\right) = \boxed{\frac{\pi\ 1088}{15}}$$

(For cylindrical method see Last Page)

(2) Rotate about $y=2$

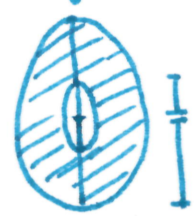


. Use x-cross Sections & Washer Method (hollow part)

. Alternative : Rotate outer curve & remove area of inner curve.

INNER : $y=\sqrt{x}$, OUTER : $y=-\sqrt{x}$

Washer



$r(x) = \sqrt{x}-2$

$R(x) = 2-(-\sqrt{x}) = 2+\sqrt{x}$

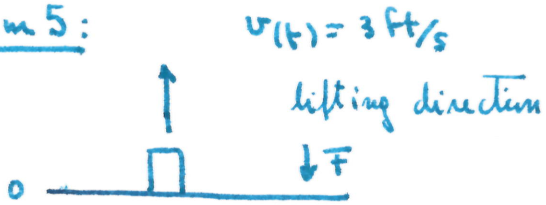
Limits of integration : $x=0$ & $x=4$

$$\text{Vol} = \int_0^4 \pi\left(R_{(x)}^2 - r_{(x)}^2\right)dx = \pi\int_0^4 \left((2+\sqrt{x})^2-(\sqrt{x}-2)^2\right)dx$$

$$= \pi\int_0^4 \left(4+x+4\sqrt{x} - (4+x-4\sqrt{x})\right)dx$$

$$= \pi\int_0^4 8\sqrt{x}\,dx = 8\pi\,\frac{2}{3}\,x^{3/2}\Big|_0^4 = \frac{16\pi}{3}\,4^{3/2} = \frac{16\pi\,8}{3} = \boxed{\frac{128\pi}{3}}$$

Problem 5:



Only force acting is weight. $F(t)$
But weight is changing with time

$F_{(0)} = 100$ lb $\qquad s(t) = 3t$ position because $s(t) = \underset{0}{s_{(0)}} + \int_0^t \underset{3}{v_{(u)}}\, du = 3t$

$F(t) = 100 - 4.5t = 100 - 4.5 \frac{3t}{3} = 100 - 4.5 \frac{s(t)}{3}$ ⇝ becomes $F(s) = 100 - 1.5s$

Total Duration $= 10$ sec ⇝ Total Trajectory

$dW = F\,ds$ $\qquad s = 0$ to $s = 3 \cdot 10 = 30$ ft

$$W = \int_0^{30} F_{(s)}\,ds = \int_0^{30} (100 - 1.5s)\,ds = 100s - \frac{3}{4}s^2 \Big|_0^{30}$$

$$= 3000 - \frac{3}{4}900 = \boxed{2325 \text{ ft-lb}}$$

Problem 6: We want the tangent line at (x_0, y_0) to be

$y = f'_{(x_0)}(x - x_0) + y_0$

& contain $(0,0)$, so $0 = f'(x_0)(0 - x_0) + y_0$ ⇝ $y_0 = x_0 f'(x_0)$

Equation becomes $\boxed{y = f'_{(x_0)}\, x}$ & $y_0 = x_0 f'(x_0)$

(1) $f(x) = e^{ax}$ so $f'(x_0) = a e^{ax_0}$

Want the line $y = a e^{ax_0} x$ to contain (x_0, e^{ax_0})

So $y_0 = e^{ax_0} = a e^{ax_0} x_0$ ⇝ $1 = a x_0$ so $\boxed{\begin{array}{l} x_0 = 1/a \\ y_0 = e^{a 1/a} = e \end{array}}$

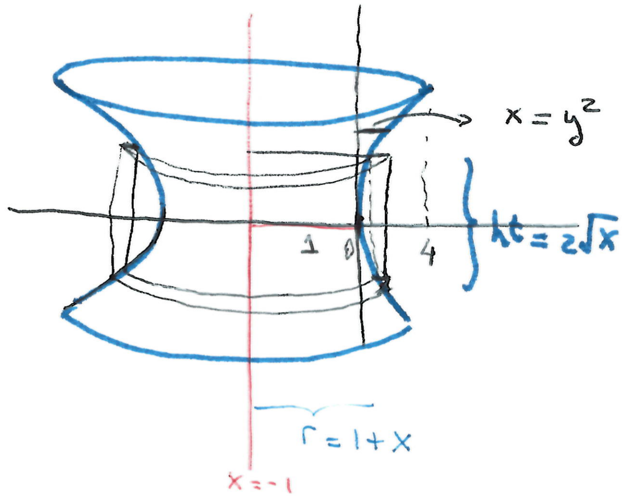
$e^{ax_0} \neq 0$

(2) $f(x) = \ln x$ so $f'(x_0) = \frac{1}{x_0}$

Want the line $y = \frac{1}{x_0} x$ to contain $(x_0, \ln x_0)$

So $\ln x_0 = \frac{1}{x_0} x_0 = 1$ gives $x_0 = e^1 = e$ so $\boxed{\begin{array}{l} x_0 = e \\ y_0 = \ln e = 1 \end{array}}$

Problem 4(1) with cylindricall shells



- Limits of integration in x = from 0 To 4
- radius = $1+x$
- height = $y_+ - y_- = \sqrt{x} - (-\sqrt{x}) = 2\sqrt{x}$

$$Vol = \int_0^4 2\pi(1+x)\,2\sqrt{x}\,dx$$

$$= 4\pi \int_0^4 (\sqrt{x} + x^{3/2})\,dx$$

$$= 4\pi \left(\frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} \Big|_0^4 \right)$$

$$= 4\pi \left(\frac{2\cdot 8\cdot 5 + 2\cdot 32\cdot 3}{15} \right) = \boxed{\frac{1088\pi}{15}}$$