

Solutions Practice Problems 3

Problem 1 We start by testing absolute convergence.

(i). Use Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{\ln(n+1)}{(n+1)^2} \right| / \left| \frac{\ln n}{n^2} \right| = \frac{\ln(n+1)}{\ln n} \cdot \frac{n^2}{(n+1)^2} \xrightarrow{n \rightarrow \infty} 1$$

$\frac{\ln(n+1)}{\ln n} \rightarrow 1$ by L'Hospital $\frac{\infty}{\infty}$ $\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\ln x} \stackrel{1}{=} \lim_{x \rightarrow \infty} \frac{1}{x+1} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{x}{x+1} = 1$

The test is inconclusive!

• Know: $\frac{\ln n}{n^2} \xrightarrow{n \rightarrow \infty} 0$ (because $\frac{\ln n}{n}$ does), so we can't rule out convergence.

• Claim: $\ln n \leq \sqrt{n}$ for n large enough

Why? We claim $\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = 0$, so for some n_0 we know

$$0 \leq \frac{\ln n}{\sqrt{n}} < 1 \quad \text{for all } n \geq n_0$$

To show the limit is what we want, we use L'Hospital $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} \sim \frac{\infty}{\infty}$

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} = 0$$

Then $\sum_{n=n_0}^{\infty} \left| \frac{\ln(n)}{n^2} \right| \leq \sum_{n=n_0}^{\infty} \left| \frac{\sqrt{n}}{n^2} \right| = \sum_{n=n_0}^{\infty} \frac{1}{n^{3/2}} < \infty$ (series for $p = 3/2 > 1$)

So by comparison, $\sum_{n=n_0}^{\infty} \left| \frac{\ln n}{n^2} \right|$ converges, and so does $\sum_{n=1}^{\infty} \left| \frac{\ln n}{n^2} \right|$

Since absolute convergence implies convergence, we are done.

Alternative: Show the alternating series converges by the AST.

Need: $\frac{\ln n}{n^2} \xrightarrow{n \rightarrow \infty} 0$ ✓

$\frac{\ln n}{n^2}$ is decreasing $\left(\frac{\ln(n+1)}{(n+1)^2} < \frac{\ln n}{n^2} \text{ for all } n \right)$

Note $f(x) = \frac{\ln x}{x^2}$ satisfies $f(x) = \frac{\ln x}{x^2}$.

$$f'(x) = \frac{\frac{1}{x} \cdot x^2 - (\ln x) 2x}{x^4} = \frac{x - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3} < 0 \text{ if } x > e^{1/2}.$$

So def. time for $x \geq 3$.

So the sequence $\frac{\ln n}{n^2}$ is decreasing for $n \geq 3$.

We use AST to show convergence!

(2) Use Root Test:

$$\sqrt[n]{\frac{e^n}{n}} = \frac{e}{\sqrt[n]{n}} \xrightarrow{n \rightarrow \infty} e > 1 \quad \text{so By Root Test not absolutely convergent}$$

$$\ln \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln a_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0 \quad \text{so } \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = e^0 = 1.$$

ln cont.

To determine convergence, need to use ~~AST~~ other techniques.

$$a_n = \frac{e^n}{n} \xrightarrow{n \rightarrow \infty} \infty, \quad \text{so the series must diverge!}$$

$(-1)^n a_n \not\rightarrow 0$

Problem 2: Need to use some algebraic manipulations.

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{5^{n/2}} &= \lim_{N \rightarrow \infty} \sum_{n=1}^N (-1)^n \frac{2^n}{(\sqrt{5})^n} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \left(\frac{(-1)2}{\sqrt{5}} \right)^n = \\ &= \lim_{N \rightarrow \infty} \left(\frac{(-1)2}{\sqrt{5}} \right) \sum_{n=0}^{N-1} \left(\frac{-2}{\sqrt{5}} \right)^n = \frac{-2}{\sqrt{5}} \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \left(\frac{-2}{\sqrt{5}} \right)^n = \frac{-2}{\sqrt{5}} \sum_{n=0}^{\infty} \left(\frac{-2}{\sqrt{5}} \right)^n \\ &= \frac{-2}{\sqrt{5}} \frac{1}{1 - \left(\frac{-2}{\sqrt{5}} \right)} = \boxed{\frac{-2}{\sqrt{5} + 2}} \end{aligned}$$

geom series with $\left| \frac{(-1)2}{\sqrt{5}} \right| < 1$

Problem 3 Write $a = e^{\ln a}$ & notice

$$a^{\ln n} = (e^{\ln a})^{\ln n} = e^{\ln a \ln n} = \underbrace{(e^{\ln a \ln n})^{\frac{1}{\ln a}}}_{a} = (e^{\ln n})^{\ln a} = n^{\ln a}$$

~~$[\frac{1}{a^{\ln n}}]$~~

So $\sum_{n=1}^{\infty} \frac{1}{a^{\ln n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\ln a}}$ so it's a p-series with $p = \ln a$

• converges for $p > 1$, meaning $\ln a > 1$
 $a > e^1 = e$ (ln incr.)

• diverges for $p \leq 1$ " $\ln a \leq 1$
 $a \leq e^1 = e$ (ln incr.)

Problem 4:

(1) Use Partial Fractions. Start by factoring the denominator:

$$x^3 + 4x = x(x^2 + 4)$$

$$\frac{4x^2 + 2x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Find A, B, C.

$$= \frac{Ax^2 + 4A + Bx^2 + Cx}{x(x^2 + 4)} = \frac{(A+B)x^2 + Cx + 4A}{x(x^2 + 4)}$$

So $A+B=4$ $\implies B=3$

$C=2$

$4A=4 \implies A=1$

$$\int_1^2 \frac{1}{x} + \frac{3x+2}{x^2+4} dx = \ln x \Big|_1^2 + \int_1^2 \frac{3x}{x^2+4} dx + \int_1^2 \frac{2}{x^2+4} dx$$

$$= \ln 2 + \frac{3}{2} \left(\ln 8 - \ln 5 \right) + \int_1^2 \frac{2}{x^2+4} dx$$

$$= \ln \left(2 \cdot \left(\frac{8}{5} \right)^{\frac{3}{2}} \right) + \int_1^2 \frac{2}{x^2+4} dx$$

$$\int_1^2 \frac{1}{x^2+4} dx = \int_1^2 \frac{1}{4\left(\left(\frac{x}{2}\right)^2+1\right)} dx = \frac{1}{4} \int_1^2 \frac{1}{\left(\frac{x}{2}\right)^2+1} dx = \frac{1}{4} \int_{\frac{1}{2}}^1 \frac{2 du}{1+u^2}$$

Substitution $\frac{x}{2} = u \quad du = \frac{dx}{2} \quad x=1 \rightarrow u = \frac{1}{2} \quad x=2 \rightarrow u = 1$

$$= \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{du}{1+u^2} = \frac{1}{2} \tan^{-1}(u) \Big|_{\frac{1}{2}}^1 = \frac{1}{2} \left(\tan^{-1}(1) - \tan^{-1}\left(\frac{1}{2}\right) \right) = \frac{1}{2} \left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2}\right) \right)$$

TOTAL: $\ln\left(\frac{16\sqrt{2}}{5\sqrt{5}}\right) + \frac{\pi}{8} - \frac{\tan^{-1}\left(\frac{1}{2}\right)}{2}$

(2) $\int_0^{\pi/4} \frac{1+\cos 2x}{\sin^2 2x} dx = \frac{1}{2} \int_0^{\pi/2} \frac{1+\cos u}{\sin^2 u} du = \frac{1}{2} \int_0^{\pi/2} \frac{1}{\sin^2 u} + \frac{\cos u}{\sin^2 u} du$
 $u=2x \quad du=2dx \quad x=0 \rightarrow u=0 \quad x=\pi/4 \rightarrow u=\pi/2$
 $= \frac{1}{2} \left(-\cot(u) + \frac{\sin u}{\sin^2 u} \right) \Big|_0^{\pi/2} = \frac{1}{2} \left(-\cot(u) + \frac{1}{\sin u} \right) \Big|_0^{\pi/2}$

$$= \frac{1}{2} \left(\frac{-(\cos u + 1)}{\sin u} \right) \Big|_0^{\pi/2} = \lim_{a \rightarrow 0^+} \frac{1}{2} \left(\frac{\cos a + 1}{\sin a} \Big|_a^{\pi/2} \right) = \frac{1}{2} (-\infty) = \boxed{\infty}$$

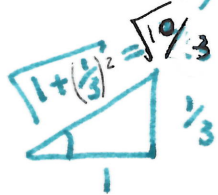
issues at 0!

$$\lim_{a \rightarrow 0^+} \frac{\cos a + 1}{\sin a} \Big|_a^{\pi/2} = \lim_{a \rightarrow 0^+} \left(\frac{0+1}{1} - \frac{\cos a + 1}{\sin a} \right) = -\infty$$

(3) $\int_1^3 \frac{dx}{\sqrt{x^2+9}} = \int_1^3 \frac{dx}{\sqrt{\left(\frac{x}{3}\right)^2+1}} = \frac{1}{3} \int_1^3 \frac{dx}{\sqrt{1+\left(\frac{x}{3}\right)^2}} = \frac{1}{3} \int_{\frac{1}{3}}^1 \frac{du}{\sqrt{1+u^2}}$
 $u = \frac{x}{3} \quad du = \frac{dx}{3}$

Use trig substitution: $u = \tan v \quad du = \sec^2 v \cdot dv \quad \sqrt{1+u^2} = \frac{1}{\cos v}$

$$= \int_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(1)} \frac{\sec^2 v}{\sec v} dv = \int_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(1)} \sec v dv = \ln(\sec v + \tan v) \Big|_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(1)}$$



$$v = \tan^{-1}(\frac{1}{3})$$

$$\sec v = \frac{1}{\cos v} = \frac{1}{\frac{1}{\sqrt{10}}} = \frac{\sqrt{10}}{1} = \frac{\sqrt{10}}{3}$$

$$\tan v = \frac{1}{3}$$

$$= -\ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right) + \ln\left(\frac{1}{\frac{\sqrt{2}}{2}} + 1\right) = \ln\left(\frac{2+\sqrt{2}}{\sqrt{2}}\right) - \ln\left(\frac{\sqrt{10}+1}{3}\right)$$



$$v = \frac{\pi}{4}$$

$$= \boxed{\ln\left(\frac{3(2+\sqrt{2})}{\sqrt{10}(\sqrt{10}+1)}\right)}$$

$$(4) \int_1^{\infty} \cos(\ln x) dx = \int_1^{\infty} \underbrace{x \frac{\cos(\ln x)}{x}}_{dv} dx = \lim_{t \rightarrow \infty} \int_1^t x \frac{\cos(\ln x)}{x} dx$$

$$v = \int \frac{\cos(\ln x)}{x} dx = \sin(\ln x) \quad (\text{Chain Rule})$$

Integration by parts gives:

$$\int_1^t x \frac{\cos \ln x}{x} dx = x \sin(\ln x) \Big|_1^t - \int_1^t \sin(\ln x) dx = (*)$$

Again

$$\int_1^t \sin(\ln x) dx = \int_1^t \underbrace{x \frac{\sin \ln x}{x}}_{dw}$$

$$w = \int \frac{\sin \ln x}{x} = -\cos(\ln x)$$

$$\int_1^t \sin(\ln x) dx = \left(-x \cos(\ln x) \Big|_1^t \right) - \int_1^t 1(-\cos \ln x) dx$$

$$(*) = x \sin(\ln x) + x \cos \ln x \Big|_1^t - \int_1^t \cos \ln x dx$$

$$= \int_1^t \cos \ln x dx$$

so. $2 \int_1^t \cos \ln x \, dx = x(\sin \ln x + \cos \ln x) \Big|_1^t$

$$\int_1^t \cos \ln x \, dx = \frac{t}{2} (\sin \ln t + \cos \ln t) - \left(\frac{\sin 0 + \cos 0}{2} \right)$$

$$= \frac{t}{2} (\sin(\ln t) + \cos(\ln t)) - \frac{1}{2}$$

Claim $\lim_{t \rightarrow 0} \frac{t}{2} (\sin(\ln t) + \cos(\ln t))$ does NO exist!

$$t = e^{n\pi} \xrightarrow[n \rightarrow \infty]{n \rightarrow \infty} \frac{t}{2} (\sin(\ln t) + \cos(\ln t)) = \frac{e^{4n\pi}}{2} (\sin 2n\pi + \cos 2n\pi)$$

$$= \frac{e^{4n\pi}}{2} (0 + (-1)^{n+1})$$

Has 2 limits $\begin{cases} n \text{ even} & \text{gives } -\infty \\ n \text{ odd} & \text{" } +\infty \end{cases}$

So the limit does exist.

Conclusion: The improper integral does NOT exist.

Problem 5:

$$(1) \lim_{x \rightarrow 0^+} \frac{\ln x}{x^2 - x} = \lim_{x \rightarrow 0^+} \frac{\ln x}{x} \cdot \frac{1}{x-1} = - \lim_{x \rightarrow 0^+} \frac{\ln x}{x} = +\infty$$

$$(2) \lim_{n \rightarrow \infty} \frac{\ln(n^{100})}{\sqrt[5]{n}} \quad \text{Enough to compute } \lim_{x \rightarrow \infty} \frac{\ln(x^{100})}{\sqrt[5]{x}} \sim \frac{\infty}{\infty} \text{ so we}$$

can use L'Hospital!

$$\lim_{x \rightarrow \infty} \frac{\ln x^{100}}{\sqrt[5]{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^{100}} \cdot 100 x^{99}}{\frac{1}{5} x^{-4/5}} = \lim_{x \rightarrow \infty} \frac{500 x^{4/5}}{x} = \lim_{x \rightarrow \infty} \frac{500}{\sqrt[5]{x}} = 0$$

$(\sqrt[5]{x})' = \frac{1}{5} x^{-4/5} \neq 0$ if $x \neq 0$

Conclusion: $\lim_{n \rightarrow \infty} \frac{\ln(n^{100})}{\sqrt[5]{n}} = 0$ as well.

(3) $\lim_{x \rightarrow 0} (\cos 2x)^{1/x^2} \sim 1^\infty$ indeterminate!

Take \ln & use L'Hospital:

$$\ln L = \ln \lim_{x \rightarrow 0} (\cos 2x)^{1/x^2} = \lim_{x \rightarrow 0} \ln ((\cos 2x)^{1/x^2}) = \lim_{x \rightarrow 0} \frac{\ln(\cos 2x)}{x^2}$$

Again $\sim \frac{0}{0}$ indeterminate so

$$\lim_{x \rightarrow 0} \frac{\ln(\cos 2x)}{x^2} = \lim_{x \rightarrow 0} \frac{-\sin 2x \cdot 2x}{2x} = \lim_{x \rightarrow 0} + \frac{\sin 2x}{2x} \left(\frac{-2}{\cos 2x} \right) = \boxed{-2}$$

$2x \neq 0$
near $x=0$ \downarrow
+1 \downarrow
-2

So $\ln L = -2$ gives $\boxed{L = e^{-2}}$.

(4) $\lim_{x \rightarrow \infty} x - \sqrt{x^2 + x}$ indeterminate $\infty - \infty$.

Mult & divide by conjugate:

$$\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x}) \frac{(x + \sqrt{x^2 + x})}{(x + \sqrt{x^2 + x})} = \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + x)}{x + \sqrt{x^2 + x}}$$

$$= \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{-1}{1 + \sqrt{1 + \frac{1}{x}}} = \boxed{\frac{-1}{2}}$$

Divide Num & Denom by x