Lecture I: §2.1 & §2.2 The problem of tangents; slopes

Courses' Textbook: Calculus with Analytic Geometry (G.F. Simmons), 2nd ed.

§2.1: What is Calculus? The problem of Tangents.

↳ "to compute"

(a) 2 Fundamental Questions:

Q1: Find the rate at which a variable quantity is changing.

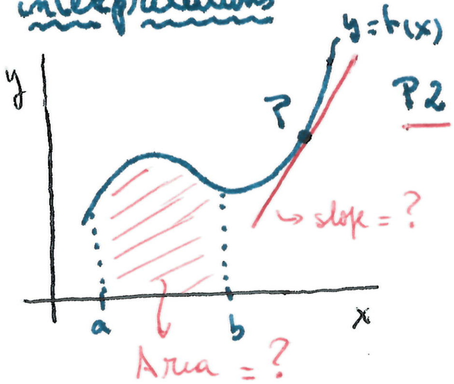
Q2: Describe a varying quantity when its rate of change is known.

("inverse problem")

Geometric interpretations: P1: Find tangent lines to curves.

($y = \boxed{m}x + b$ m = slope ; $y = f(x)$)

P2: Find the area under a graph.



P1: Differential Calculus (Ch 2 – 5)

P2: Integral Calculus (Ch 6 – 12)

"Fundamental" Thm of Calculus (§6.6)

(Part III: Sequences & Infinite Series)

(b) Why?

- From computations going back to Archimedes & Greeks to its formalization by
  - Newton (1642 – 1727)
  - Leibniz (1646 – 1716)
- Part of the basic language of science; to describe continuous motions

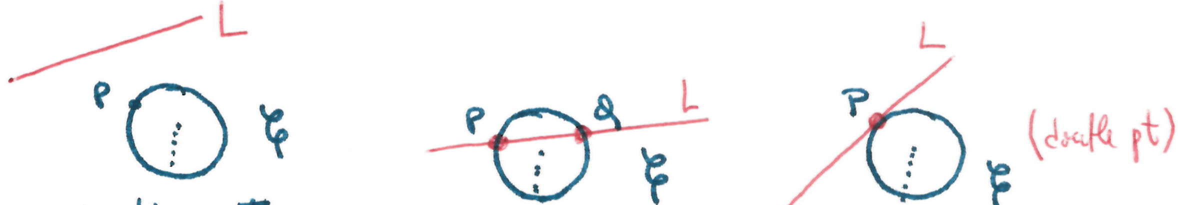
eg. motions of planets + gravity (eg discovery of Neptune)

- Biology: Hodgkins-Huxley eqn describing the action potential across neurons in the brain.
- Economics: Black-Scholes eqn. modelling option pricing in financial markets.

- Why formalize & axiomatize? Concentrate on the underlying structure of different phenomena ⇝ gain flexibility by means of abstraction

(c) What is a Tangent line? "tangible = to touch"

- Idea 1: Look at lines relative to curves (eg circles)



- Don't meet
- $\mathcal{C}$ on ONE SIDE

- Meet at 2 pts P & Q
- $\mathcal{C}$ on TWO SIDES

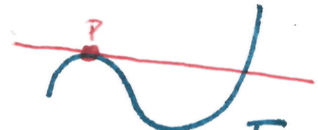
(Secant Line)

- Meet at 1 pt P
- $\mathcal{C}$ on ONE Side

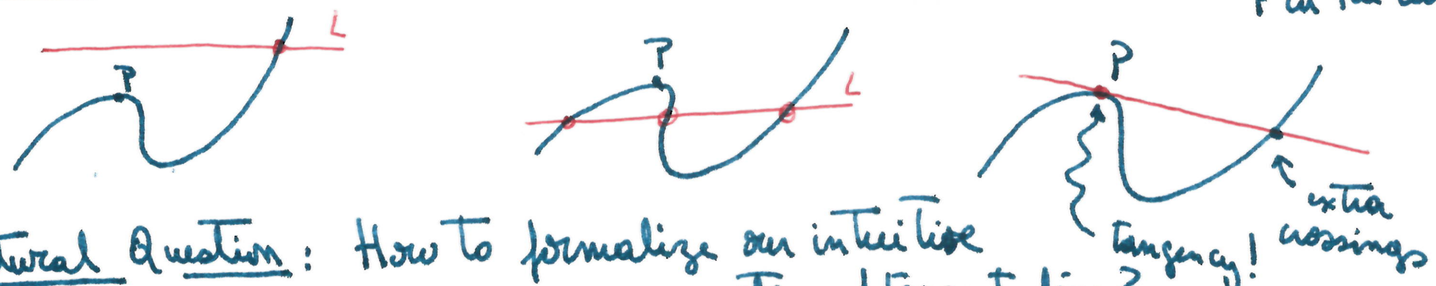
(Tangent Line)

A definition via "counting the number of intersection points" is too restrictive & it fails for general curves.

- Idea 2: Curve on "one side" of the line or on both & meeting at only one point also fails in general



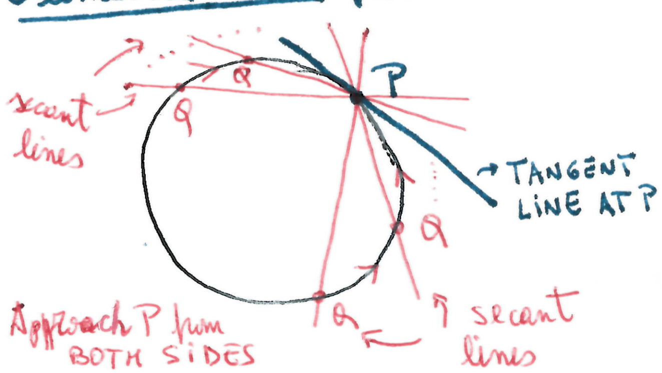
Solution: Combine these 2 ideas but in a LOCAL situation (around a pt P in the curve)



Natural Question: How to formalize our intuitive notion of tangent line?

A (Fermat ~1630) "Tangent lines are LIMITS of secant lines"

Geometric description:



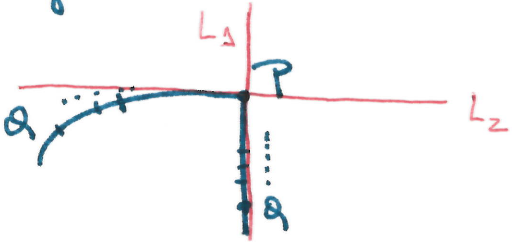
Fermat's idea:

1. Fix P & a second pt Q on the curve
2. Draw the (secant) line through P & Q.
3. Move Q towards P & see how the secant line moves along with Q.
4. Provided the Tang line at P exists, these secant lines will approach this limiting Tang line

Remark: Almost all of Calculus involves some limiting process.

⚠ Tangent lines don't always exist!

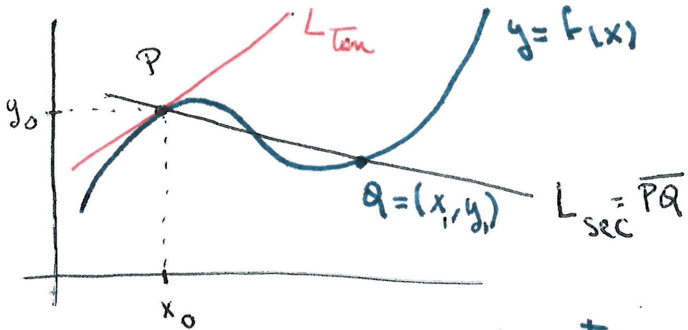
Eg.



Q: Which one is the tangent? L_1 or L_2?

A: Neither! There is no Tang line at P.

§2.2 How to calculate the slope of the tangent?



Recall A line through $P = (x_0, y_0)$ has equation $\boxed{y = m(x - x_0) + y_0}$

m = slope = only parameter to be determined

- m is obtain via a limiting process" $\Rightarrow$ we need coordinates!
- Eqn of secant line $L_{sec} = \overline{PQ}$: $y = m_{sec}(x - x_0) + y_0$

Where $\boxed{m_{sec} = \dfrac{y_1 - y_0}{x_1 - x_0}}$ $P = (x_0, y_0)$, $Q = (x_1, y_1)$

- Eqn of tangent line L_{Tan}: $y = m_{Tan}(x - x_0) + y_0$

Since $\underset{m_{Tan}}{\underset{\updownarrow}{L_{Tan}}} = \lim\limits_{Q \to P} \underset{m_{sec}}{\underset{\updownarrow}{L_{sec}}}$, then $m_{Tan} = \lim\limits_{Q \to P} m_{sec}$ $(Q \neq P)$

$$\boxed{m_{Tan} = \lim_{\substack{x_1 \to x_0 \\ y_1 \to y_0 \\ (x_1 \neq x_0,\ y_1 \neq y_0)}} \frac{y_1 - y_0}{x_1 - x_0}}$$

Next Time: Concrete examples.