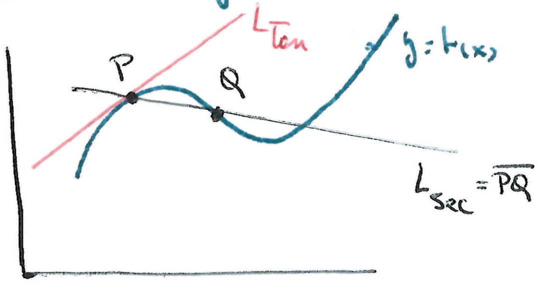
Lecture II : § 2.2 (cont) Slopes of tangents
§ 2.3 : Definition of derivative

Recall: "Tangent lines are limits of secant lines"



Write $P=(x_0,y_0)$

$Q=(x_1,y_1)$ approaches P

$L_{Tan}: y = m_{Tan}(x-x_0)+y_0$

$L_{sec} \quad y = \dfrac{y_1-y_0}{x_1-x_0}(x-x_0)+y_0$

m_{sec}

$$\boxed{m_{Tan} = \lim_{\substack{x_1\to x_0\\ y_1\to y_0}} \frac{y_1-y_0}{x_1-x_0}}$$ Slope of tangent line at P.

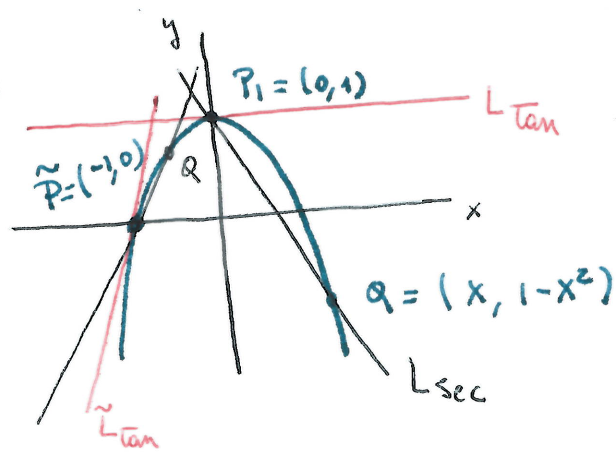
Important fact: We ALWAYS have $Q \neq P$, so if the curve is the graph of a function $y=f(x)$, we have
- $x_1 \neq x_0$ ("vertical line test")
- $y_0 = f(x_0)$ & $y_1 = f(x_1)$

So $$\boxed{m_{Tan} = \lim_{\substack{x_1\to x_0\\ [f(x_1)\to f(x_0)]}} \frac{f(x_1)-f(x_0)}{x_1-x_0}} \quad \left(\overset{=:}{\text{def.}}\ f'(x_0)\right)$$

[Usually $f(x_1)\to f(x_0)$ will be a consequence of $x_1\to x_0$ for "nice" functions f.]

Some Numerical examples:

$y = 1-x^2 \ (= f(x))$



Ex 1: $P=(0,1)$, $x_0=0,\ y_0=1$

$$m_{Tan} = \lim_{x\to 0}\frac{(1-x^2)-1}{x-0} = \lim_{x\to 0}\frac{-x^2}{x} \overset{\boxed{x\neq 0}}{=} \lim_{x\to 0} -x = \boxed{0}$$ ⇝ horizontal!

$\boxed{L_{Tan}: y = 1}$

Ex 2: $\tilde{P}=(-1,0)$, $x_0=-1 \quad y_0=0$

$$m_{Tan} = \lim_{x\to -1}\frac{(1-x^2)-0}{x-(-1)} = \lim_{x\to -1}\frac{1-x^2}{x+1} = \lim_{x\to -1}\frac{(1-x)(1+x)}{1+x}$$

$$\overset{\boxed{x+1\neq 0}}{=} \lim_{x\to -1} 1-x = 1-(-1) = \boxed{2}$$, so $\boxed{\tilde{L}_{Tan}: y = 2x+2}$

Conclusions
- Geometry tells us what the tangent should be
- "Calculus" (or Analysis) allows us to formally compute it

(How? Choose the pt P; compute m_{tan} as a limit; write down L_{tan})

Formal procedure: "Method of increments"

- Think of $P=(x_0,y_0)$ as being fixed
- Move the point $Q=(x_1,y_1)$ towards P : $\boxed{\begin{aligned} x_1 &= x_0+\Delta x \\ y_1 &= y_0+\Delta y \end{aligned}}$

where Δx & Δy are small increments (positive or negative!) that we let go to zero.

$$\rightsquigarrow \quad m_{sec} = \frac{y_1-y_0}{x_1-x_0} = \frac{\Delta y}{\Delta x} \quad \text{so} \quad m_{tan} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

Special case: $y=f(x)$, so $y_0=f(x_0)$, $y_1=f(x_1)$

Substitution gives: $\Delta y = y_1-y_0 = f(x_1)-f(x_0) = f(x_0+\Delta x)-f(x_0)$

Conclusion:
$$\boxed{\begin{aligned} m_{sec} &= \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \\ m_{tan} &= \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \end{aligned}}$$

Earlier examples: $y=1-x^2$, $P=(0,1)$, $\tilde{P}=(-1,0)$

$$m_{sec} = \frac{1-(x_0+\Delta x)^2-(1-x_0^2)}{\Delta x} = \frac{-(x_0^2+(\Delta x)^2+2x_0\Delta x)+x_0^2}{\Delta x}$$
$$= \frac{-\Delta x\,(\Delta x+2x_0)}{\Delta x} \underset{\boxed{\Delta x\neq 0}}{=} \boxed{-(\Delta x+2x_0)}$$

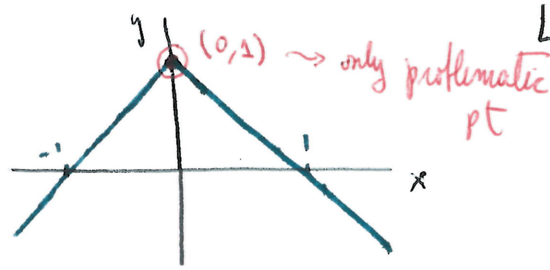
Ex 1: $m_{sec} = -(\Delta x+2\cdot 0) = -\Delta x \xrightarrow[\Delta x\to 0]{} 0 = m_{tan}$ ✓

Ex 2: $m_{sec} = -(\Delta x+2\cdot(-1)) = 2-\Delta x \xrightarrow[\Delta x\to 0]{} 2 = m_{tan}$ ✓

Same answers as before ☺

- <u>An example with no tangent</u>

$$y = f(x) = 1 - |x| = \begin{cases} 1-x & \text{if } x \geq 0 \\ 1+x & \text{if } x<0 \end{cases}$$



All pts <u>except</u> $(0,1)$ admit a tangent

- Increment method gives $y = x+1$ for $P=(x_0,y_0)$ with $x_0<0$ (slope 1)
- ———————— $y = 1-x$ ———————— $x_0>0$ (slope -1)

[Also: curve is piecewise linear with corner locus $= (0,1)$]

- At $P = (0,1)$ $\quad m_{\tan} = \lim\limits_{\Delta x \to 0} \dfrac{(1-|0+\Delta x|)-1}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{-|\Delta x|}{\Delta x} = \begin{cases} -1 & \text{if } \Delta x > 0 \\ 1 & \text{if } \Delta x < 0 \end{cases}$

so the limit does <u>not</u> exist! (sidewise limits are different!)

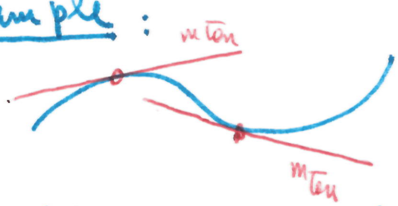
§2.3 <u>Derivative of a function</u>:

(1) What is a function?

<u>Def</u>: A <u>function</u> $f: D \to \mathbb{R}$ defined on a set of real numbers D is a formula / rule / law of correspondence that assigns a <u>single</u> real number y to each number x in D.

- Write $y = f(x)$ or $x \longmapsto f(x)$
- Call D = domain of f
- <u>Range of f</u> $= \mathrm{Im}(f) =$ the values y we get assigned by f.
- x = independent variable
  y = dependent variable

<u>Example</u>:



To each P, we assign $m_{\tan}$ = slope of the tangent line at P

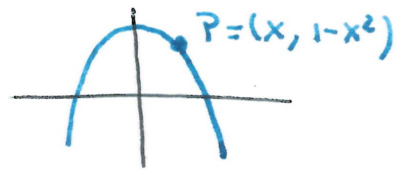
If we use coordinates & the curve is $y = f(x)$

we define a "<u>slope</u>" <u>function</u> $\quad x_0 \longmapsto \lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x} =: f'(x)$

- Call it: the <u>derivative</u> of $f(x)$

Note: Since tangent lines not always exists, the domain of f' might be smaller than the domain of f.

Example 1 $f_{(x)} = 1 - x^2$ Domain $= \mathbb{R}$



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1-(x+\Delta x)^2 - (1-x^2)}{\Delta x}$$

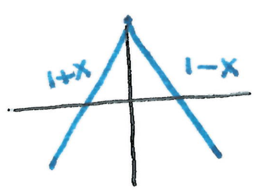
$$\underset{\substack{\uparrow \\ \text{earlier calculation}}}{=} \lim_{\Delta x \to 0} (-2x - \Delta x) = -2x$$

So f' is defined everywhere & its formula is $f'(x) = -2x$

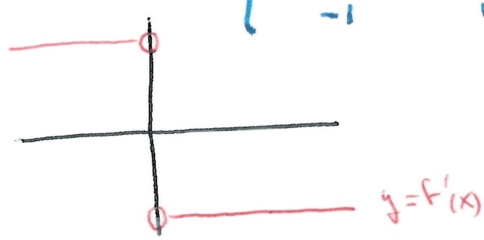
Check numerical examples: $x = 0$: $f'(0) = 0$ ✓

$x = -1$: $f'(-1) = -2(-1) = 2$ ✓

Example 2 $f(x) = 1 - |x| \rightsquigarrow f'(x) = \begin{cases} 1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x > 0 \end{cases}$ Domain: $\{x \neq 0\}$



Graph of f



Graph of f'

Notation:

- Leibniz: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

- $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} = \frac{d}{dx} f$

Think: "Operation performed on the function f"

Correct notation can clarify concepts!

Exercises: Compute $f'(x)$ for $f(x) = x^3$, $f(x) = \frac{1}{x}$ $(x \neq 0)$ & $f(x) = \sqrt{x}$ $(x \geq 0)$