Lecture III §2.5 The concept of a limit.

Recall: Given a function $f: D \to \mathbb{R}$ where D is a subset of $\mathbb{R}$ (write $D \subseteq \mathbb{R}$)
we define f' as a new function with formula

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

whenever the limit for x in D exists.

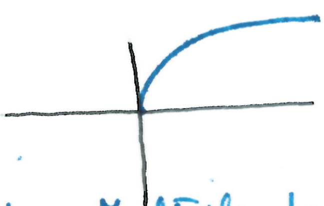
Eg ① If $f(x) = x^2$, then $f'(x) = 2x$ & it's defined everywhere (last time)

② $f(x) = \frac{1}{x}$ (for $x \neq 0$)

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x - (x+\Delta x)}{x(x+\Delta x)\Delta x} = \lim_{\Delta x \to 0} \frac{-\cancel{\Delta x}}{x(x+\Delta x)\cancel{\Delta x}} = -\frac{1}{x^2}.$$

again defined only where $x \neq 0$.

③ $f(x) = \sqrt{x}$ for $D = \{x \geq 0\}$ $\rightsquigarrow$ $f'(x) = \frac{1}{2\sqrt{x}}$ for $x > 0$ (not $x = 0$!)



Why? $f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$

Trick: Multiply by $1 = \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$

So $$f'(x) = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \begin{cases} \text{no limit } (\infty) & x = 0 \\ \frac{1}{2\sqrt{x}} & x \neq 0 \end{cases}$$

Notation:

- Leibniz: $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} = \frac{d}{dx} f$$

  Δf: increment in f. (small!)
  Δx: incr in x (small)
  THINK: "Operation performed on the function f"

- $f'(x_0) = \left.\frac{df}{dx}\right|_{x=x_0}$ (when evaluating at the point x_0)

$\boxed{\text{Correct notation can clarify concepts!}}$

So far: we have USED limits, but never really define them precisely...
(Only relying on our intuition!)

Q: What does $\lim_{x \to a} g(x) = L$ mean?

Ingredients:
- A function $g : D \to \mathbb{R}$ defined around a, not necessarily at a
- A number L

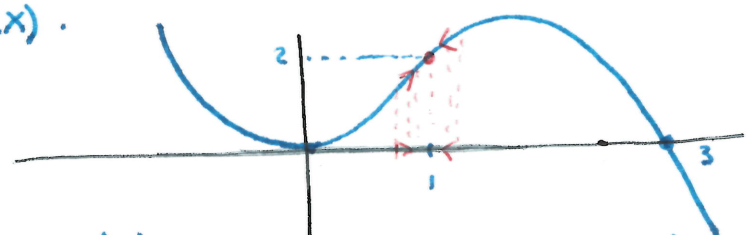
Ideas (*) As x approaches a, the value of $g(x)$ approaches L

(**) We can make $g(x)$ be as close as we want (not very precise!) to L by taking x close enough to a.

Note: We are not claiming anything about the value $g(a)$ (and we shouldn't care about it!) [it may not be possible to evaluate g at a]

The way to understand this notion of limit depends on the way we understand the function $g(x)$.

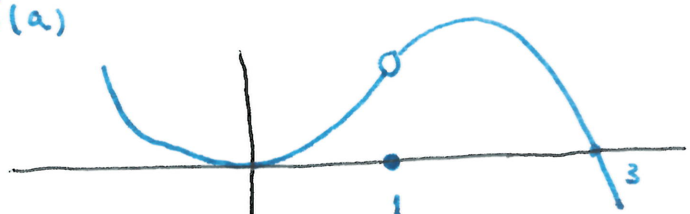
Ex 1 $g(x) = 3x^2 - x^3$



Can guess the limit $\lim_{x \to 1} g(x) = \lim_{\Delta x \to 0} g(1 + \Delta x) = g(1) = 2$ from the graph.

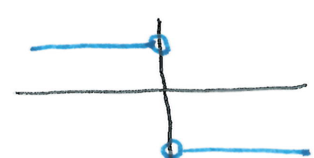
Same for other values: $\lim_{x \to a} g(x) = g(a)$

Ex 2 $g(x) = \begin{cases} 3x^2 - x^3 & \text{for } x \neq 1 \\ 0 & \text{for } x = 1 \end{cases}$



Still $\lim_{x \to 1} g(x) = 2 \quad (\neq g(1))$ (because away from $x = 1$, both pictures agree!)

Ex 3 $f(x) = \dfrac{-x}{|x|}$ for $x \neq 0$ $= \begin{cases} -1 & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases}$



$\lim_{x \to 1} f(x) = -1$ but $\lim_{x \to 0} f(x)$ does not exist!

Q: What can we say if we can't draw the graph of g?

A We make (**) more precise $\rightsquigarrow$ $\boxed{\text{GAME OF CHOICE}}$

- Pick how close to L <u>YOU</u> want to be (say, for example, $|f(x)-L| < 10^{-8}$) & challenge <u>ME</u> to determine how close to a must x be so if say $0 < |x-a| < \boxed{10^{-26}}$ then for <u>all</u> these x we can <u>ENSURE</u> $|f(x)-L| < 10^{-8}$.

my pick

<u>Definition</u>: We say $\lim_{x\to a} f(x) = L$ if for <u>every</u> $\varepsilon > 0$ (<u>your</u> choice) there is a $\delta > 0$ (<u>my</u> choice) such that <u>if</u> $0 < |x-a| < \delta$ <u>then</u> $|f(x) - L| < \varepsilon$

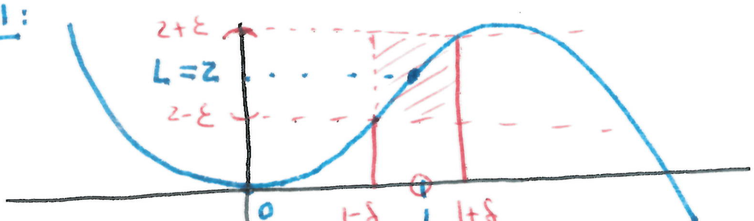
(rules out $x=a$)

(in short: I <u>always</u> win the game of choice)

<u>Typically</u>: δ has a formula in terms of ε and a.

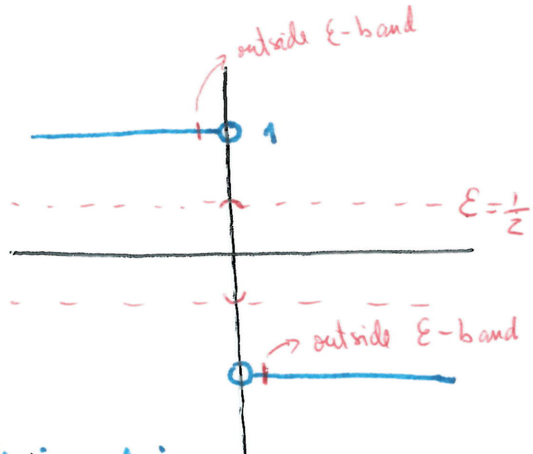
- Even if we have a formula for f, finding δ given ε and a can be very challenging (because it involves "inverting" a formula)
- How to find δ from a graph?

<u>Ex 1</u>:



graph of $f : (1-\delta, 1+\delta) \to \mathbb{R}$ lies within this ε-horizontal strip [IGNORING $x=1$!]

<u>Ex 2</u>:



<u>Claim</u>: $\lim_{x\to 0} \frac{-|x|}{x} \neq 0$

Why? Pick $\varepsilon = \frac{1}{2}$.

For <u>any</u> $\delta > 0$, $f\left(\frac{\delta}{2}\right) = -1$

so for $x = \frac{\delta}{2}$: $|x - 0| = \frac{\delta}{2} < \delta$ <u>BUT</u>

$\frac{1}{2} < |f(x) - 0| = |f(\frac{\delta}{2})| = |-1| = 1$

- Similar thing happens for <u>any</u> L in $\mathbb{R}$

<u>Example</u>: $g(x) = 1 - x^2$ Prove $\lim_{x \to 0} f(x) = 1 = L$

By $\varepsilon - \delta$: given any $\varepsilon > 0$, we want to find $\delta > 0$ (in terms of ε) so that if $0 < |x - 0| = |x| < \delta$, then $|g(x) - 1| = |1 - x^2 - 1|$
$= |-x^2| = |x^2| < \varepsilon$

But if $|x| < \delta$, then $x^2 < \delta^2$

If we pick $\delta^2 = \varepsilon$, then $x^2 < \delta^2 = \varepsilon$ as we wanted everytime $0 < |x| < \delta$.

So $\delta = \sqrt{\varepsilon}$ does the trick!