Lecture IV: § 2.5 Two trigonometric limits
§ 2.4 Rates of change & velocity

§1 More on limits

Properties (Limit laws) Assume $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} g(x) = M$. Then

(1) $\lim_{x\to a} f(x) \pm g(x) = L \pm M$

(2) $\lim_{x\to a} f(x)\, g(x) = LM$

(3) For any real number c, $\lim_{x\to a} c f(x) = cL$

(4) If $M \neq 0$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$

Proofs Next Time (Appendix A2)

• Two trigonometric limits:

Ex 1: $\lim_{h\to 0} \dfrac{\sin h}{h} = ?$

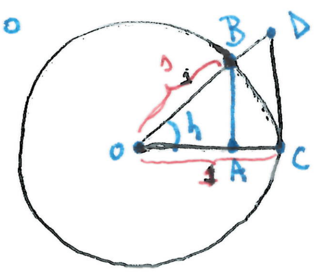
sin(h) 

Note: $\sin(0) = 0$ so we have a $\frac{0}{0}$ - indeterminacy

Q: Do things cancel out, like $c\dfrac{x^2}{x}$, $c\dfrac{x}{x^2}$ or $\dfrac{cx}{x}$?

• Use trigonometry in the unit circle

Say $h>0$



$\dfrac{AB}{1} = \sin(h)$

$\dfrac{CD}{1} = \tan(h)$

$\dfrac{OA}{1} = \cos(h)$

Ingredients

• arc of circ = BC

• sector OBC of the circle

• 2 triangles $\overset{\triangle}{OAB}$ & $\overset{\triangle}{OCD}$

Area of $\overset{\triangle}{OBC}$ = Area circle · fraction of the circle $= \pi \cdot \dfrac{h}{2\pi} = \dfrac{h}{2}$ $\left(\dfrac{|h|}{2} \text{ if } h<0\right)$

Area of $\overset{\triangle}{OAB} = \dfrac{1}{2}$ base · height $= \dfrac{1}{2} OA \cdot AB = \dfrac{\cos(h)}{2}\sin(h)$ $\left(\dfrac{\cos h}{2}\sin|h| \text{ if } h<0\right)$

Area of $\overset{\triangle}{OCD}$ = ______ $= \dfrac{1}{2} OC \cdot CD = \dfrac{1}{2}\tan(h) = \dfrac{\sin(h)}{2\cos(h)}$

So Area of $\overset{\triangle}{OAB} \leq$ Area sector OBC $\leq$ Area $\overset{\triangle}{OCD}$ $\left(\dfrac{\sin|h|}{2\cos h} \text{ if } h<0\right)$

$$\boxed{\dfrac{\cos(h)\sin(h)}{2} \leq \dfrac{h}{2} \leq \dfrac{\sin(h)}{2\cos h}}$$

$\dfrac{-\cos h \sin h}{2} = \dfrac{\cos h \sin|h|}{2} \leq \dfrac{|h|}{2}$ if $h<0$ $\leq \dfrac{\sin(|h|)}{2\cos h} = \dfrac{-\sin h}{2\cos h}$

• If $h>0$, divide by $\frac{h}{2}$ everywhere

$$\cos h \ \frac{\operatorname{sen} h}{h} \le 1 \le \frac{1}{\cos h}\ \frac{\operatorname{sen} h}{h}$$

so $\cos h \le \frac{\operatorname{sen} h}{h} \le \frac{1}{\cos h}$ $\quad$ ($\cos(0)=1$ so $\cos(h)>0$ for h near 0)

• If $h<0$, divide by $\left(-\frac{h}{2}\right)>0$ everywhere

$$\frac{+\cos h \operatorname{sen} h}{2h} \le 1 \le \frac{1}{\cos h}\ \frac{\operatorname{sen} h}{h} \quad \text{so} \quad \cos h \le \frac{\operatorname{sen} h}{h} \le \frac{1}{\cos h}$$

again.

. We know $\cos h \le \frac{\operatorname{sen} h}{h} \le \frac{1}{\cos h}$ & $\lim_{h\to 0} \cos h = 1 = \lim_{h\to 0} \frac{1}{\cos h}$

So $\frac{\sin h}{h}$ is squeezed between 1 & 1 in the limit.

Conclusion: $\boxed{\lim_{h\to 0} \frac{\operatorname{sen} h}{h} = 1}$

Application: $\lim_{h\to 0} \frac{\cos h - 1}{h} = ?$ (also $0/0$ - indet.)

both have a limit!

$$\frac{\cos h - 1}{h}\ \frac{\cos h + 1}{\cos h + 1} = \frac{\cos^2 h - 1}{h(\cos h + 1)} \underset{\boxed{\operatorname{sen}^2 h + \cos^2 h = 1}}{=} \frac{-\operatorname{sen}^2 h}{h(\cos h + 1)} = \underbrace{\frac{\operatorname{sen} h}{h}}_{\downarrow h\to 0 \ \ 1} \cdot \underbrace{\left(\frac{-\operatorname{sen} h}{1+\cos h}\right)}_{\downarrow \ \frac{-0}{2} = 0}$$

So $\boxed{\lim_{h\to 0} \frac{\cos h - 1}{h} = 0}$

§ 2.4 Velocity & rates of change

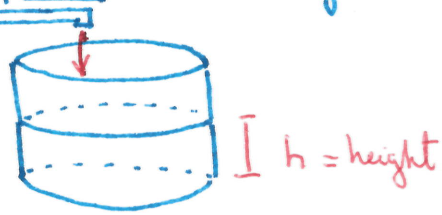
Think of $y = f(x)$ as indicating a relationship between 2 physical quantities

$$\frac{\Delta y}{\Delta x} \ \text{average rate of change} \ \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{\leadsto} f' = \frac{dy}{dx} \ \text{instantaneous rate of change}$$

[So Problem of tangent slopes becomes the Problem of rate of change]

Applications 1: The independent variable $x = t$ is TIME

Example ① Filling a water tank



$V(t)$ = Volume (of water) at time t

$\frac{dV}{dt}$ = rate at which the tank is filled

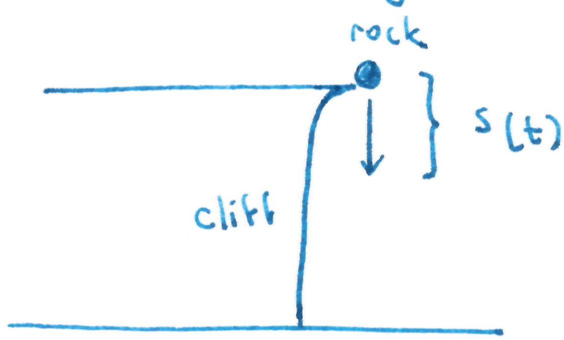
Also: the height $h(t)$ is changing with time

$\frac{dh}{dt}$ = rate at which the height changes

Note: V & h are related $\quad V(t) = h(t) \cdot \overbrace{\text{Area of the base}}^{\text{fixed!}}$

(Example: If $h(t) = t$, then $\frac{dV}{dt} = \frac{d}{dt}(t \cdot \text{Area}) = \underbrace{\text{Area of the base}}_{\text{fixed number}}$)

Example ②: Rock falling of a cliff:



$s(t)$ = position at time t

- $\frac{ds}{dt} = s'(t) := v(t)$ velocity

(It has a "direction": pos or neg)

- $\frac{dv}{dt} = v'(t) =: a(t)$ acceleration

$v(t)$ = rate of change of position

$a(t)$ = ———— velocity

Speed $= |v(t)|$ no direction! ⟿ dashboard of car.

Here: experimental results propose $\quad s(t) = 16t^2 \quad (ft)$

$s'(t) = 32t \quad$ (prove it by definition)

[acceleration from GRAVITY] ← $a(t) = 32 \quad (ft/s^2) \quad (9.8 \, \frac{m}{s^2})$

Basis of Newtonian mechanics.

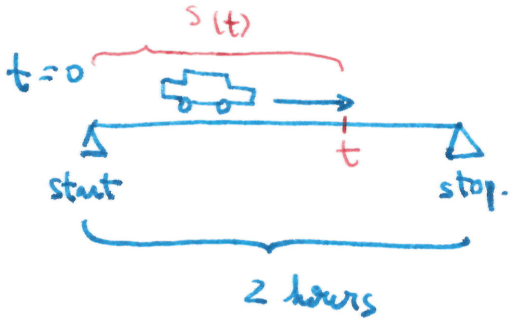
Applications 2: The indep. variable can be something else, $x = \#$ cars produced

$C(x)$ = cost of producing x cars

$\dfrac{dC}{dx}$ = marginal cost or "cost per unit" (we don't make fractional cars. 1 car = instantan…

↳ Econ or Business term for rate of change

Application 3: Car driving straight down the road



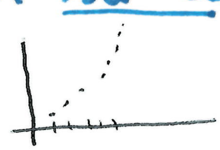
Odometer = measures distance travelled

Speedometer = ―― speed (forward velocity)

- Watching odometer gives $s(t)$ = distance at time t
- ―― speedometer ―― $v(t) = s'(t)$ = velocity at time t

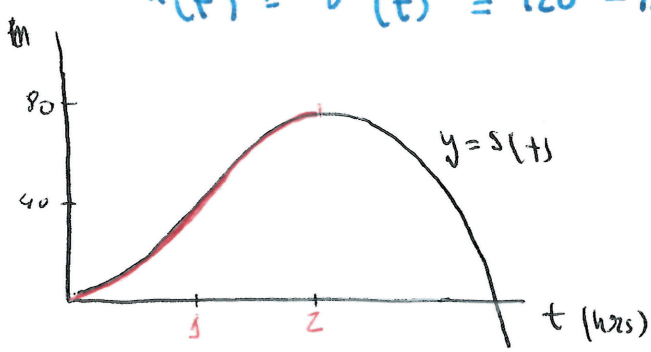
By empirical measure, one might find or estimate that

$$s(t) = 60t^2 - 20t^3$$

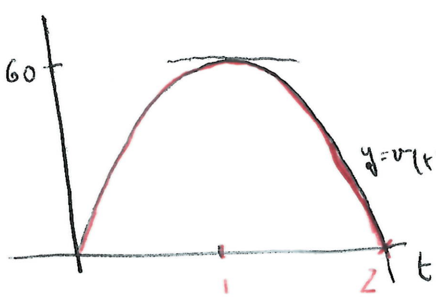


so $v(t) = s'(t) = 120t - 60t^2$ ⇝ check with speedometer

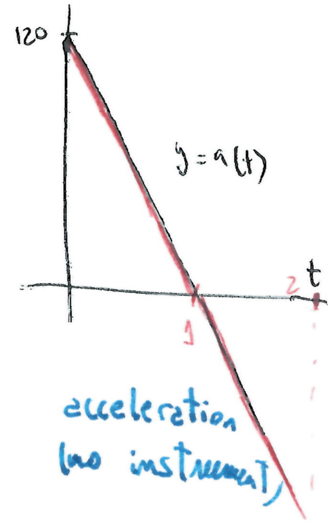
$$a(t) = v'(t) = 120 - 120t = 120(1-t)$$



position
(ODOMETER)



velocity
(SPEEDOMETER)



acceleration
(no instrument)

We stop measuring after $t = 2$ hours

- Each graph gives a "different" picture of car travels.