Lecture V Appendix A2 Theorems about limits

Recall: Given a function $f: D \to \mathbb{R}$ defined around $x = a$ (not neces. at a)

we say $\lim_{x\to a} f(x) = L$ (in $\mathbb{R}$) if for EVERY $\varepsilon > 0$ (your choice), we can find $\delta > 0$ (depending on ε & a) (MY choice) such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

("So the limit is L if I win the game of choice ALWAYS!")

Ex: $f(x) = 5x+4$ $\quad a = 0 \quad$ Guess: $\lim_{x\to 0} f(x) = 4$

WANT: $|(5x+4)-4| = |5x| = 5|x| < \varepsilon$ if $|x| < \delta$.

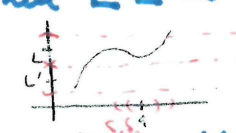
So we back-track our steps: if $\left.\begin{array}{l}|x| < \delta \\ 5|x| < 5\delta\end{array}\right\}$ taking $5\delta \leq \varepsilon$ will do the trick! eg $\delta = \frac{\varepsilon}{5}$

Harder examples: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (last time) ; $\lim_{x\to 0} (1+x)^{1/x} = e \simeq 2.7182$ (Ch. 8)

(We'll use other tricks that follow from the definition of limit)

- Natural Question 1: Can we approach 2 different limits L, L'?
- Natural Question 2: How do limits behave with respect to the four standard operations $(+, -, \cdot, /)$ in $\mathbb{R}$? What about inequalities?

Thm 1 If $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} f(x) = L'$, then $L = L'$.



Proof: Say $L' \neq L$, pick $\varepsilon = |L - L'| > 0$.

By def: we can find $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \varepsilon/2$

we can find $\delta_2 > 0$ so that if $0 < |x-a| < \delta_2$, then $|f(x) - L'| < \varepsilon/2$

Pick $\delta = \min\{\delta_1, \delta_2\} > 0$ & assume $0 < |x-a| < \delta$.

Then $\varepsilon = |L - L'| = |(L - f(x)) + (f(x) - L')| \underset{\Delta \text{ ineq}}{\overset{|c+d| \leq |c|+|d|}{\leq}} \underbrace{|L - f(x)|}_{<\frac{\varepsilon}{2}} + \underbrace{|f(x) - L'|}_{<\frac{\varepsilon}{2}} < \frac{2\varepsilon}{2} = \varepsilon$

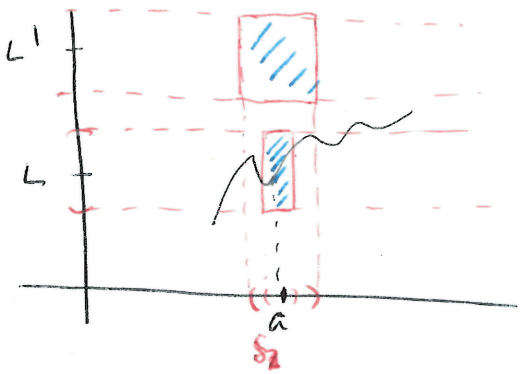
so we get $\varepsilon < \varepsilon$, but this is NOT possible!

So what went wrong? A: Our original assumption $L \neq L'$ leads to a contradiction, so it must be FALSE. We conclude $\boxed{L = L'}$ ∎

Note: This is an example of a **Proof by Contradiction**

- Δ-inequalities $|c+d| \le |c| + |d|$ will be very useful.
- To combine inequalities, useful to take $\delta = \min\{\delta_1, \delta_2, \dots\}$.

Alternative proof: Use a picture



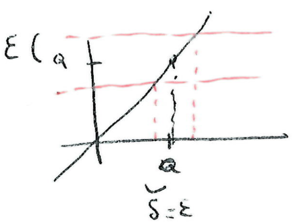
δ_1 works for L
δ_2 " " L' $\Big\}$ $\delta = \min\{\delta_1, \delta_2\}$

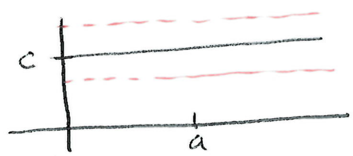
If we pick $0 < \varepsilon = |L - L'|$, & $0 < |x-a| < \delta$, then $f(x)$ must lie in 2 "boxes" with no points in common! This is impossible!

Warm-up limits:

Thm 2 $\lim_{x \to a} x = a$ & $\lim_{x \to a} c = c$ for any constant c.

Proof:
- Want $0 < |x-a| < \varepsilon$ if $0 < |x-a| < \delta$. Pick $\delta = \varepsilon$.
- Want $0 = |\overset{f(x)}{c} - \overset{L}{c}| < \varepsilon$ if $0 < |x-a| < \delta$. Pick **ANY** $\delta > 0$, for example $\delta = 1$.





Always within the strip!

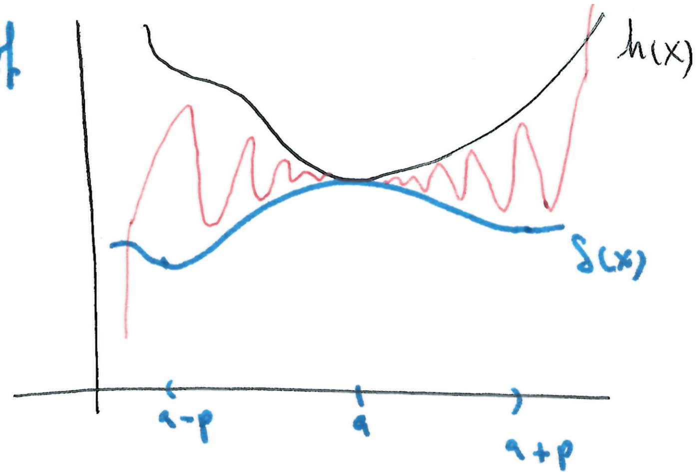
TOMORROW: Limit laws via ε/δ.

Squeeze Thm Assume $g(x) \le f(x) \le h(x)$ in a neighborhood of a (that is, if $0 < |x-a| < p$ for some p)

If $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then the limit $\lim_{x \to a} f(x)$ exists & its value is also L

Recall: we used this last time to show $\lim_{x \to 0} \frac{\sin x}{x} = 1$ with $g(x) = \cos(x)$, $h(x) = \frac{1}{\cos(x)}$.

Proof



- We use ε/δ for $f(x)$.

  Fix $\varepsilon > 0$.

- Pick $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then $|h(x) - L| < \varepsilon$

- Pick $\delta_2 > 0$ so that if $0 < |x-a| < \delta_2$, then $|g(x) - L| < \varepsilon$

But $|h(x) - L| < \varepsilon$ means $L - \varepsilon < h(x) < L + \varepsilon$

$|g(x) - L| < \varepsilon$ —— $L - \varepsilon < g(x) < L + \varepsilon$

Pick $\delta = \min\{p, \delta_1, \delta_2\} > 0$ & assume $0 < |x - a| < \delta$.

Then:

$$L - \varepsilon \underset{\delta \le \delta_1}{<} g(x) \underset{\delta \le p}{\le} f(x) \underset{\delta \le p}{\le} h(x) \underset{\delta \le \delta_2}{<} L + \varepsilon$$

so by looking at the ends, we get

$$L - \varepsilon < f(x) < L + \varepsilon.$$

But this is the same as $|f(x) - L| < \varepsilon$, which is what we needed to show $\lim_{x \to a} f(x) = L$ by definition. $\square$