Lecture VI: Appendix A2 (cont) More theorems about limits

§2.6 cont functions, intermediate value Theorem

§1 Limit laws:

Thm 1 (Limit laws by ε/δ) If $\lim\limits_{x\to a} f(x) = L$ & $\lim\limits_{x\to a} g(x) = M$, then

1) $\lim\limits_{x\to a} f(x) \pm g(x) = L \pm M$

2) $\lim\limits_{x\to a} f(x)\, g(x) = L \cdot M$

Proof (1) Pick any $\varepsilon>0$.

By def of L: we can find $\delta_1>0$ so that if $0<|x-a|<\delta_1$, then $|f(x)-L|<\varepsilon/2$

—— M: —— $\delta_2>0$ —— $0<|x-a|<\delta_2$, — $|g(x)-M|<\varepsilon/2$

Take $\delta = \min\{\delta_1,\delta_2\}$ & assume $0<|x-a|<\delta$.

Want to show: $|f(x)+g(x)-(L+M)|<\varepsilon$

- $|f(x)+g(x)-(L+M)| \underset{\uparrow \text{ rearrange}}{=} |f(x)-L+g(x)-M| \leq \underbrace{|f(x)-L|}_{\substack{<\varepsilon/2\\ (\delta\leq\delta_1)}} + \underbrace{|g(x)-M|}_{\substack{<\varepsilon/2\\ (\delta<\delta_2)}} < 2\frac{\varepsilon}{2} = \varepsilon$

as we wanted!

- Proof for $f(x)-g(x)$ ~~is~~ the ~~same~~.

(2) Pick $\varepsilon>0$. Start from what we want.

$|f(x)g(x)-LM| \underset{\substack{\uparrow\\ +\,\&\,-\, f(x)M}}{=} |f(x)g(x)-f(x)M+f(x)M-LM| = |f(x)(g(x)-M)+(f(x)-L)M|$

$\leq |f(x)|\,|g(x)-M| + |f(x)-L|\,|M| \underset{\substack{\uparrow\\ |M|\leq|M|+1}}{\leq} |f(x)|\,|g(x)-M|+|f(x)-L|\,(|M|+1)$

↳ could be 0

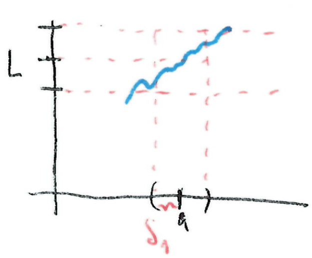
Q: Can we approximate each summand? Want to find $\delta>0$ so that

: $|f(x)|\,|g(x)-M| < \varepsilon/2$ & $(|M|+1)\,|f(x)-L| < \frac{\varepsilon}{2}$ if $0<|x-a|<\delta$

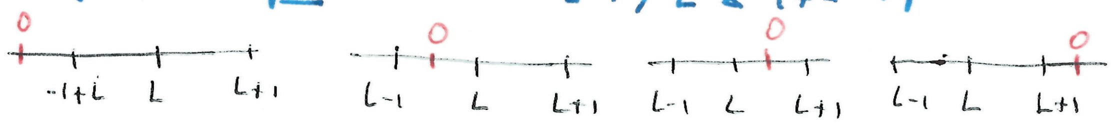
- First summand: has $|f(x)|$ & $|g(x)-M|$

  - For $f(x)$: Take $\tilde{\varepsilon}=1$. we can find $\delta_1>0$ so that if $0<|x-a|<\delta$, then

$|f(x)-L|<1$ that is: $-1+L<f(x)<1+L$, Then $\boxed{|f(x)|<1+|L|}$ (*)



(*) case-by-case analysis: Look at $L-1$, L & $1+L$ & place 0



$|L+1| > |L-1|$, $\quad |L+1| = 1+L = 1+|L|$

$|L-1|<|L+1|$, $\quad 1-L = 1+|L|$

$|L-1| > |L+1|$, $\quad = 1+|L|$

$|L-1| > |L+1|$, $\quad = 1+|L|$

• for $g(x)$: find $\delta_2 > 0$ so that if $0 < |x-a| < \delta_2$ then

$|g(x) - M| < \frac{\varepsilon}{2(1+|L|)}$ $\quad (\tilde{\varepsilon} = \frac{\varepsilon}{2(1+|L|)} > 0)$

If $\delta \le \min\{\delta_1, \delta_2\}$, we get $|f(x)|\,|g(x) - M| < (1+|L|)\frac{\varepsilon}{2(1+|L|)} = \varepsilon$ if $0<|x-a|<\delta$.

• Second summand: we can find $\delta_3 > 0$ so that if $0 < |x-a| < \delta_3$, then

$|f(x) - L| < \frac{\varepsilon}{2(1+|M|)}$ $\quad (\tilde{\varepsilon} = \frac{\varepsilon}{2(1+|M|)} > 0)$.

If $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, both summands $< \frac{\varepsilon}{2}$, so their sum $< \varepsilon$ □.

Consequence: Any polynomial $f = c_n x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0$ with real coefficients $c_n, c_{n-1}, \dots, c_0$ satisfies $\lim_{x \to a} f(x) = c_n a^n + \dots + c_1 a + c_0 = f(a)$.

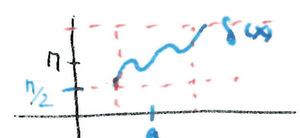
. These are our favorite continuous functions!

Thm 2: If $\lim_{x \to a} g(x) = M \neq 0$, then $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$.

Proof: Write $\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{g(x)\,M}\right| = |g(x) - M| \frac{1}{|M|} \frac{1}{|g(x)|}$ $\quad (*)$

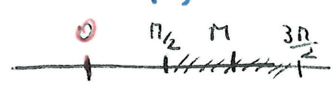
Look at the 2 factors separately:

• For $\frac{1}{g(x)}$: since $M \neq 0$, we know $|M| > 0$.

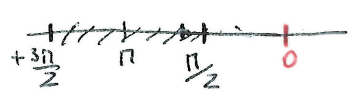


We can find $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then $|g(x) - M| < \tilde{\varepsilon} = \frac{|M|}{2}$

so $M - \frac{|M|}{2} < g(x) < \frac{|M|}{2} + M$



$M > 0$



$M < 0$

In both cases $0 < \frac{|M|}{2} < |g(x)| < \frac{3|M|}{2}$

so $\boxed{\frac{2}{|M|} > \frac{1}{|g(x)|}} > \frac{2}{3|M|}$ $\rightsquigarrow$ $\left|\frac{1}{g(x)} - \frac{1}{M}\right| < |g(x) - M| \frac{1}{|M|} \frac{2}{|M|}$

• Need $\delta_2 > 0$ so that $|g(x) - M| < \frac{\varepsilon |M|^2}{2}$ if $0 < |x-a| < \delta_2$.

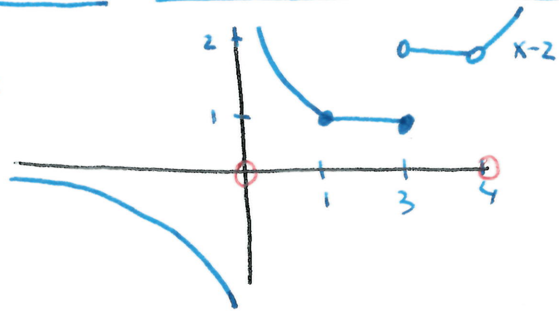
If $\delta = \min\{\delta_1, \delta_2\}$, then $\left|\frac{1}{g(x)} - M\right| < \frac{\varepsilon |M|^2}{2} \frac{1}{|M|} \frac{2}{|M|} = \varepsilon$ if $0<|x-a|<\delta$.

Consequence If $\lim_{x\to a} g(x) = M \neq 0$ & $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

Why? Write $\frac{f(x)}{g(x)} = f(x)\,\frac{1}{g(x)}$ & use product rule.

§2 Continuous functions & the mean value Theorem:

Example



$$f(x) = \begin{cases} \frac{1}{x} & x<0 \\ \frac{1}{x} & 0<x<1 \\ 1 & 1\leq x\leq 3 \\ 2 & 3<x<4 \\ x-2 & x>4 \end{cases}$$

Domain $D = \mathbb{R} \setminus \{0,4\} = (x\neq 0,4)$.

- f is cont. everywhere on D except at $x=3$ because $\left.\begin{array}{l}\lim_{x\to 3^+} f(x) = 2 \neq f(3) \\ \lim_{x\to 3^-} f(x) = 1 = f(3)\end{array}\right\}$ no limit!
- We can extend f to $x=4$ in a continuous way by declaring $f(4) = \lim_{x\to 4} f(x) = 2$
- We can't ———— $x=0$ ———— since $\lim_{x\to 0} f(x)$ does not exist

Def: A function f defined in a neighborhood of a pt a is continuous at a if
1) f is defined at a
2) $\lim_{x\to a} f(x)$ exists & $= f(a)$

A function f is continuous if it's continuous for every pt in its domain.

Note $\lim_{x\to a} f(x) = f(a)$ is the same as $\lim_{x\to a} (f(x) - f(a)) = 0$.

Using increments: $\lim_{\Delta x\to 0} \underbrace{f(a+\Delta x) - f(a)}_{=\Delta f} = 0$.

Prop: If $f'(a)$ exists, (f differentiable at $x=a$) then f MUST be continuous at a.

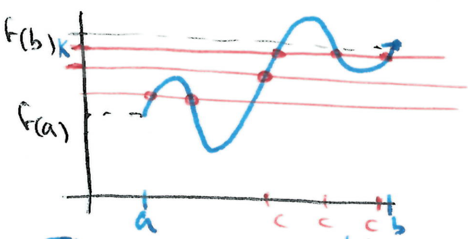
Proof: $\lim_{\Delta x\to 0} \Delta f = \lim_{\Delta x\to 0} \underbrace{\frac{\Delta f}{\Delta x}}_{f'(a)} \underbrace{\Delta x}_{0} \underset{\text{Prod rule}}{=} f'(a)\cdot 0 = 0$. □

Remark: The other implication fails (f cont at $a \not\Rightarrow f'(a)$ exists.)

For example $f(x) = |x|$ cont at $x=0$, but $f'(0)$ does not exist

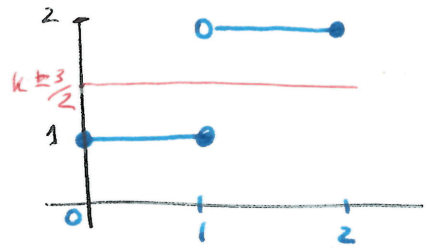
§3: The Intermediate Value Theorem

IVT: If $f:[a,b] \longrightarrow \mathbb{R}$ is cont, then every K between $f(a)$ & $f(b)$ is **attained**, meaning we can find c in $[a,b]$ with $K=f(c)$.



The graph crosses the horiz line $Y=K$ at least once

vs



The result fails for $K=3/2$ because f is not cont at 1

Special case: If $f(a)>0$ & $f(b)<0$ (or viceversa) & f is cont, then we can find c in $[a,b]$ with $f(c)=0$

Example: $f(t) = 3t^2+t^3+1$ — ↑ dominant term

$\lim_{t\to\infty} f(t) = +\infty$

$\lim_{t\to-\infty} f(t) = -\infty$

so it has a real root!

$f(0) = 0+0+1 = 1 > 0$

$f(-4) = 3\cdot 16 - 64 + 1 = 48-64+1 = -15<0$ } we have a real root in between -4 & 1

We can refine our search by bisecting our interval.

say $f(-2) > 0 \rightsquigarrow$ root in $[-4,-2] \rightsquigarrow f(-3) = 1 > 0 \rightsquigarrow$ root in $[-4,-3]$

$\rightsquigarrow$ sg $f(-3.5) = ?$, etc.