Lecture VII §2.6 (cont.) The Mean Value & the Extreme Value Thms

§1 The Mean Value Thm

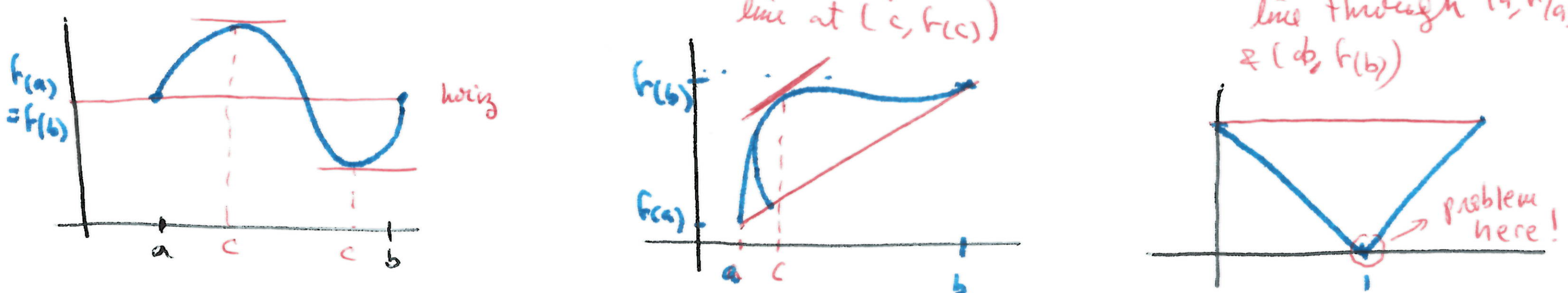
MVT: Assume $f : [a,b] \longrightarrow \mathbb{R}$ satisfies

1) f is cont on $[a,b]$ {meaning $f(a) = \lim_{x \to a^+} f(x)$, $f(b) = \lim_{x \to b^-} f(x)$ & cont inside.)
2) f is diff'ble on (a,b) (exclude $x = a, b$)

Then, we can find c in (a,b) where $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

slope of tangent line at $(c, f(c))$

slope of the secant line through $(a, f(a))$ & $(b, f(b))$



$f(a) = f(b)$
2 values for c
["Rolle's Thm"]

1 value for c
("horiz line" from different viewpoint)

NON-example
$f(x) = |x-1|$
$a = 0$, $b = 2$ $f(0) = f(2) = 1$
cont but not diff'ble (at $x=1$)

Proof: See Appendix A4 (future lecture)

Application: We can use the sign of f' to predict growth behavior of f.

In the following, assume f is cont on $[a,b]$ & diff'ble on (a,b):

Consequence 1: If $f'(x) > 0$ everywhere on (a,b), then f is strictly increasing on (a,b) (meaning if $a < s < t < b$ then $f(s) < f(t)$)

Consequence 2: If $f'(x) < 0$ everywhere on (a,b) then f is str. decreasing on (a,b)

Consequence 3: If $f'(x) = 0$ everywhere on (a,b) then f is constant.

Why? Restrict f to $[s,t]$, so f is cont. on $[s,t]$ & diff'ble on (s,t).

By MVT we can find c with $f'(c) = \dfrac{f(t) - f(s)}{t-s}$ so $\underbrace{(t-s)}_{>0} f'(c) = f(t) - f(s)$

so $\text{sgn}(f'(c)) = \text{sign of } (f(t) - f(s))$

$+ \rightsquigarrow f(t) > f(s)$ for all $s < t$ so Conseq 1 ✓

$- \rightsquigarrow f(t) < f(s)$ Conseq 2 ✓

$0 \rightsquigarrow f(t) = f(s)$ for all t, s so f is constant! Conseq 3 ✓

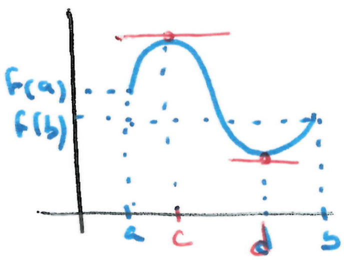
§ 2 The Extreme Value Theorem

EVT If f is cont on $[a,b]$, then f attains both a maximum & a minimum value in $[a,b]$ ("the extreme values")

Proof: 1) Need to show the function is bounded, ie we can find M & N with
$$M \leq f(x) \leq N \quad \text{for all } x \text{ in } [a,b]$$

2) Can adjust M & N to be the best lower & upper bounds. [Appendix A3] (future lecture)

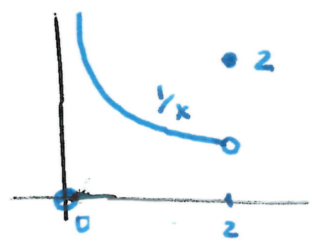
3) Show these optimal M & N bounds are achieved.



$f(c)$ is MAX

$f(d)$ is MIN

[Here: f is diff'ble at (a,b) & $f'(c)=f'(d)=0$]

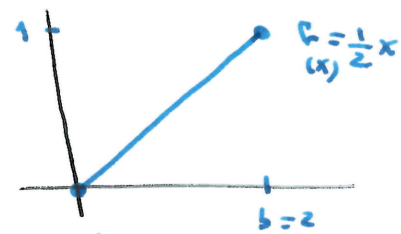


$\lim\limits_{x\to 0^-} f(x) = +\infty$

{ no min attained
no max attained

$f: (0,2] \to \mathbb{R}$

not cont.



$f(a)$ is MIN

$f(b)$ is MAX

f is diff'ble BUT

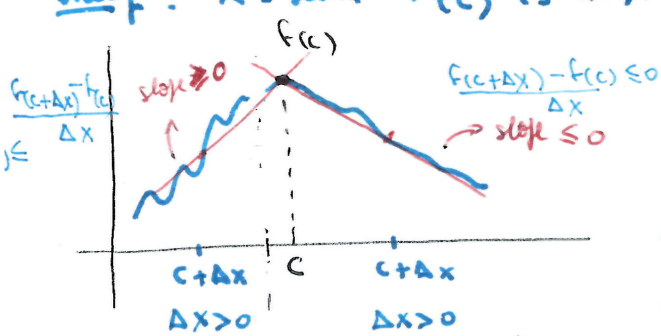
$f'(a) \neq 0$, $f'(b) \neq 0$

[a, b are boundary pts]

⚠ Important to consider the boundary pts. Also: don't need f to be diff'ble but it helps find MAX & MIN if it is!

Consequence: If c in (a,b) is an extreme value for f & f is cont. on $[a,b]$ & f is diff'ble at c, then $f'(c)=0$ (horiz tangent line at $(c, f(c))$).

Proof: Assume $f(c)$ is MAX (if MIN use similar reasoning)



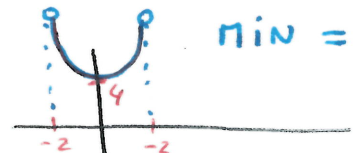
- Use: slope of secants between $(c, f(c))$ & $(c+\Delta x, f(c+\Delta x))$ $\longrightarrow$ slope of tangent at $(c, f(c))$
- On right: secant slopes are ≤ 0 so $f'(c) \leq 0$
- " left: ——— ≥ 0 so $f'(c) \geq 0$

Only conclusion possible: $\boxed{f'(c)=0}$ □

Application 1: Show that $f(x) = 4 + x^2$ has max & min values on $[-2,2]$ & find them. What about at $(-2,2)$?

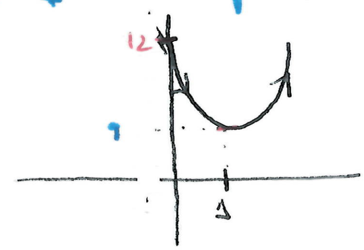
Soln: f is continuous on $[-2,2]$ & so by EVT we have a max & min. Since f is also diff'ble, we can find them as follows:

1) Find $f'(x) = 0$ so $2x = 0 \rightsquigarrow \boxed{x=0}$ $\quad f(0) = 4$
2) Compare with end-pts $f(-2) = 8$, $f(2) = 8$
3) Pick largest value & smallest value. MAX $= 8$ at $x = -2, 2$; MIN $= 4$ " $x = 0$.



• No max on $(-2,2)$ but min because $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^-} f(x) = 8$ was the max.

Application 2: Assume $s(t) = 12 - 6t + 3t^2$ is the position at time t of an object moving on a straight line. Find velocity & acceleration & decide if at any point the object changes direction.



Soln: $v(t) = s'(t) = -6 + 6t = 6(t-1)$

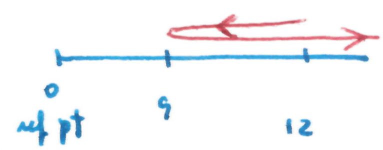
$a(t) = v'(t) = 6 \cdot 1 = 6$ constant!

• To change direction means $v(t)$ changes sign! So need to find its zeros

$v(t) = 0$ for $t = 1$. If $t < 1$ $v(t) < 0$ ($s(t)$ is decreasing)

If $t > 1$ $v(t) > 0$ (— increasing)

This is the vertex of the parabola ($s(1) = 12 - 6 + 3 = 9$)



In these 2 examples we used derivatives of polynomials rules. We'll see today they work next time.