Lecture VIII §3.1 Derivatives of polynomials
§3.2 Product & Quotient Rules

§1: Rules of derivation of $f(x) = c_n x^{n} + \dots + c_2 x^2 + c_1 x + c_0$ (polynomial of degree n with real coeffs)

Prop (1) $\frac{d}{dx} c = 0$ (derivative of a constant function)

(2) $\frac{d}{dx}(x^n) = n x^{n-1}$ for any positive integer n

Proof: By definition!

1) $c(x) = c$ for all x so $\frac{dc}{dx} = \lim_{\Delta x \to 0} \frac{\overset{=c}{c(x+\Delta x)} - \overset{=c}{c(x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$

2) want to show: $\lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = n x^{n-1}$.

To do so, we need to rewrite the numerator $(a+b)^n = \underbrace{(a+b)(a+b)\cdots(a+b)}_{n \text{ times}}$
(so that we can cancel Δx!)

Use distribution to get

Binomial Thm:
$$(a+b)^n = a^n + n a^{n-1} b + \overset{(*)}{\frac{n(n-1)}{2}} a^{n-2} b + \dots + \overset{k^{th}\text{ term}}{\boxed{\frac{n(n-1)\cdots(n-k)}{1\,2\cdots k}}} a^{n-k} b^k + \dots + n a b^{n-1} + b^n$$
$$= a^n + n a^{n-1} b + \binom{n}{2} a^{n-2} b + \dots + \boxed{\binom{n}{k}} a^{n-k} b^k + \dots + \binom{n}{n-1} a b^{n-1} + b^n.$$

palindromic in a & b with integer coefficients. "n choose k"

Example: $(a+b)^2 = a^2 + \overset{\binom{2}{1}}{2} ab + b^2$ $\qquad (a+b)^3 = a^3 + \overset{\binom{3}{1}}{3} a^2 b + \overset{\binom{3}{2}}{3} a b^2 + b^3$, etc.

In our case: $(x+\Delta x)^n = x^n + n x^{n-1} \Delta x + \dots + \binom{n}{k} x^{n-k} (\Delta x)^k + \dots + n x (\Delta x)^{n-1} + (\Delta x)^n$

so
$$\frac{(x+\Delta x)^n - x^n}{\Delta x} = n x^{n-1} \frac{\cancel{\Delta x}^{\,1}}{\cancel{\Delta x}_{\,1}} + \dots + \binom{n}{k} x^{n-k} \frac{(\Delta x)^{\cancel{k}\, k-1}}{\cancel{\Delta x}} + \dots + n x \frac{(\Delta x)^{\cancel{n-1}\, n-2}}{\cancel{\Delta x}} + \frac{(\Delta x)^{\cancel{n}\, n-1}}{\cancel{\Delta x}}$$

Each summand $\binom{n}{k} x^{n-k} (\Delta x)^{k-1} \xrightarrow[\Delta x \to 0]{} 0$ if $\overset{k \geq 1}{k \geq 2}$

so only $k=1$ term survives when taking $\lim_{\Delta x \to 0}$. This was $n x^{n-1}$, as we wanted! □

(*) Choose 2 slots for b in an ordered fashion
n choices for 1st b
$n-1$ " " 2nd b } overcount by ordering so we need to divide by 2

(E.g. think $n=3$ & choose 2 boxes for 2 colored balls).

General Properties: (1) $\frac{d}{dx}(c f_{(x)}) = c \frac{df}{dx}$

(2) $\frac{d}{dx}(f_{(x)} + g_{(x)}) = \frac{df}{dx} + \frac{dg}{dx}$

These follow DIRECTLY from the definition of derivatives + limit laws.
(exercise)

Consequence: If $f = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial of degree $n \geq 0$

then $\frac{df}{dx} = a_n n x^{n-1} + \cdots + \underbrace{a_k k x^{k-1}}_{\text{(general term)}} + \cdots + a_2 2x + a_1$.

Q: How to interpret these 2 general properties?

A: "Differentiation is a linear operator on the space of functions"
$f \longmapsto f'$

Linear
- sends sums to sums (2)
- sends multiplication by constants (scalars) to multiplication by constants (1).

This linear condition & spaces with these 2 operations are the subject of LINEAR ALGEBRA (MATH 2568)

§2. Product Rule:

Thm 1: Assume $f_{(x)}, g_{(x)}$ are differentiable, then the product $h_{(x)} = f_{(x)} g_{(x)}$ is also differentiable at x & $\frac{d}{dx}(f_{(x)} g_{(x)}) = \frac{df}{dx}(x)\, g_{(x)} + f_{(x)} \frac{dg}{dx}(x)$

Proof. Use definition!

$$\frac{f(x+\Delta x)\, g(x+\Delta x) - f(x)\, g(x)}{\Delta x} \overset{+\,\&\,-\, f(x)\, g(x+\Delta x)}{=} \frac{f(x+\Delta x)\, g(x+\Delta x) - f(x)\, g(x+\Delta x) + f(x)\, g(x+\Delta x) - f(x)\, g(x)}{\Delta x}$$

$$\overset{\text{rearrange}}{=} \underbrace{\frac{(f(x+\Delta x) - f(x))}{\Delta x}}_{\downarrow \Delta x \to 0 \; f'(x)} \underbrace{g(x+\Delta x)}_{\downarrow \Delta x \to 0 \; g(x)\ \text{(diff'l, so cont.)}} + f(x) \underbrace{\frac{(g(x+\Delta x) - g(x))}{\Delta x}}_{\downarrow \Delta x \to 0 \; g'(x)} \xrightarrow[\Delta x \to 0]{} f'g + fg'$$

(Limit Laws)

□

Example 1: Verify that this agrees w/ Power Rule.

$$4x^3 \underset{\text{Power Rule}}{=} \frac{d}{dx}(x^4) = \frac{d}{dx}(x^3 x) = \boxed{\frac{dx^3}{dx}}\, x + x^3 \frac{dx}{dx} = \boxed{3x^2}\, x + x^3 = 4x^3$$

$$\frac{dx^3}{dx} = \frac{d(x^2 x)}{dx} = \underbrace{\frac{dx^2}{dx}}_{=2x \text{ (again)}} x + x^2 1.$$

Example 2 $f = (2x-5)(x^3-4x+8)$

- Use distribution: $f = 2x^4 - 5x^3 - 8x^2 + 36x - 40 \rightsquigarrow f'(x) = 8x^3 - 15x^2 - 16x + 36$
- Use Prod Rule: $f' = 2(x^3-4x+8) + (2x-5)(3x^2-4) = 2x^3 - 8x + 16 + 6x^3 - 15x^2 - 8x - 20$ (same!)

§2. Quotient Rule:

Q: Given $f(x)$ & $g(x)$, when can we define $h(x) = \frac{f(x)}{g(x)}$?

A Need $g(x) \neq 0$.

Domain of $h = \{x : f \,\&\, g$ are both defined at x & $g(x) \neq 0\}$
(in Domain f & Domain of g)

How to differentiate $h(x)$? Write $h(x) = f(x) \frac{1}{g(x)}$ & use product rule

$$h'(x) = f' \frac{1}{g} + f \boxed{\left(\frac{1}{g}\right)'}$$

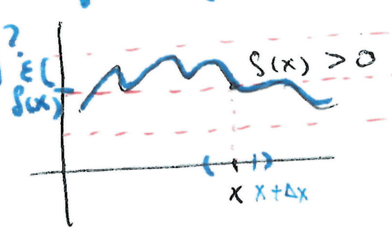
How to do this?

Thm 2: Assume $g(x) \neq 0$ & g is differentiable at x. Then $\rho(x) = \frac{1}{g(x)}$ is defined in a neighborhood of x & it's differentiable at x with

$$\rho'(x) = \frac{-g'(x)}{g^2(x)}.$$

Proof: Since g is diff'ble at x, it's continuous at x. We can show that the sign of g is constant in a neighborhood of x, in particular it's never 0.

Why?



Say $g(x) > 0$, then for $\varepsilon = \frac{g(x)}{2}$ we can find $\delta > 0$ so that if $0 < |\Delta x| < \delta$, then $|g(x+\Delta x) - g(x)| < \varepsilon$

$$0 < \frac{g(x)}{2} = g(x) - \varepsilon < g(x+\Delta x) < \varepsilon + g(x) = \frac{3g(x)}{2}$$

In particular: $\rho(x+\Delta x)$ is defined if $0 < |\Delta x| < \delta$, so we can check if ρ is differentiable at x.

$$\frac{P(x+\Delta x)-P(x)}{\Delta x} = \frac{\frac{1}{g(x+\Delta x)}-\frac{1}{g(x)}}{\Delta x} = \frac{g(x)-g(x+\Delta x)}{\underbrace{g(x)\,g(x+\Delta x)}\,\Delta x} \xrightarrow[\Delta x\to 0]{\text{[Limit Laws]}} -\frac{g'(x)}{g^2(x)}$$

$\downarrow \Delta x \to 0$ $g(x)$ $\quad \downarrow \Delta x\to 0$ $g(x)$ $\quad \Delta x \to 0$ $-g'(x)$

Consequence: Power rule with negative exponents!

Eg $f(x) = x^{-3} = \frac{1}{x^3}$ $= g(x)$ $\rightsquigarrow f'(x) = \frac{-g'(x)}{g^2(x)} = \frac{-3x^2}{(x^3)^2} = -3x^{-4}$

In general: $f(x) = x^{-n} = \frac{1}{x^n} \rightsquigarrow f'(x) = \frac{-nx^{n-1}}{(x^n)^2} = -\frac{nx^{n-1}}{x^{2n}} = -n x^{n-1-2n}$
$= -n x^{-n-1}$.

Quotient Rule: $\left(\frac{f}{g}\right)' = f'\frac{1}{g} + f\left(\frac{1}{g}\right)' = \frac{f'}{g} + f\left(\frac{-g'}{g^2}\right) = \frac{f'g - fg'}{g^2}$.

Example 3: Decide where $f = \frac{x+1}{x-1}$ is defined,
" continuous
" diff'ble.

Soln: Only Need $x-1 \neq 0$ so Domain $f = (x \neq 1)$.

- f is continuous for all $x \neq 1$ (quotient of cont func is continuous when the denominator doesn't vanish).

Note $\lim_{x\to 1^+} \frac{x+1}{x-1} = +\infty$ ($x-1>0$) & $\lim_{x\to 1^-}\frac{x+1}{x-1} = -\infty$ ($x-1<0$) so we can't extend continuously to $x=1$

- Use quotient rule $f'(x) = \frac{\overset{1}{(x+1)'}(x-1) - (x+1)\overset{1}{(x-1)'}}{(x-1)^2} = \frac{(x-1)-(x+1)}{(x-1)^2}$
$= \frac{2x}{(x-1)^2}$

In general Quotients of polynomials $\frac{P(x)}{Q(x)}$ are called rational functions

- They are defined everywhere except when $Q(x)=0$ (zeroes of the polynomial)
- —— continuous on their domain because $\lim_{x\to a}\frac{P(x)}{Q(x)} = \frac{\lim_{x\to a}P(x)}{\lim_{x\to a}Q(x)} = \frac{P(a)}{\boxed{Q(a)}}$ $\neq 0$ for a.
- —— differentiable —— because of the quotient rule!