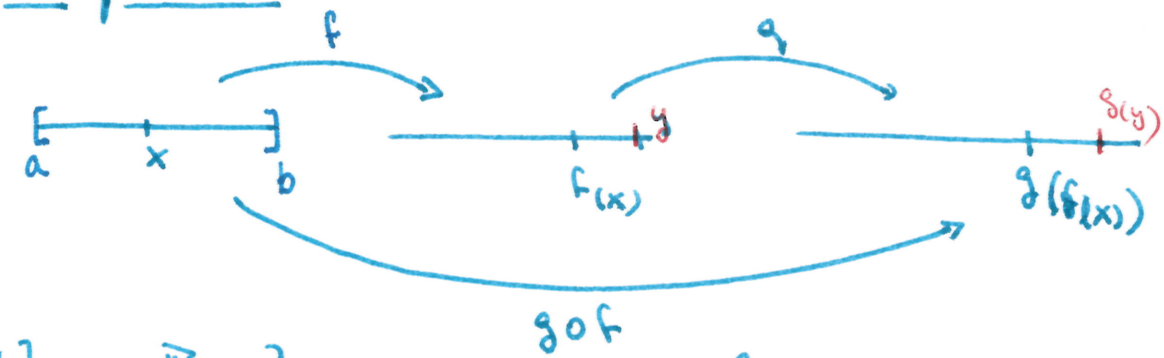
# Lecture IX § 3.3 Composite functions & the Chain Rule

## §1 Composite functions

IDEA




$f : [a,b] \longrightarrow \mathbb{R}$
$g : \mathbb{R} \longrightarrow \mathbb{R}$ } $\rightsquigarrow$ Composite $\boxed{g \circ f}$ is a new function on $[a,b]$ defined as $x \longmapsto g(f(x))$

[ First apply $f$ & THEN apply $g$ to $f(x)$ ]

GOAL Want to see how nice properties of $f$ & $g$ pass onto $g \circ f$ (if they do at all!) (Eg: Continuity & derivatives)

**THM 1:** If $f$ is continuous at $x_0$ & $g$ is continuous at $y_0 = f(x_0)$, then $g \circ f$ is continuous at $x_0$.

**THM 2** (Chain Rule) If $f$ is differentiable at $x_0$ & $g$ is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at $x$ and $(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0)$

Easy way to remember this: "increment notation"

$$\frac{d\, g \circ f}{dx} = \frac{dg}{df}\,\frac{df}{dx} \quad \text{meaning} \quad \left.\frac{dg}{dy}\right|_{y=f(x)} \cdot \frac{df}{dx}.$$

**Example:** $f(x) = (x^3 + 4x)$ $\quad g(y) = y^{10}$ $\rightsquigarrow$ $g \circ f(x) = (x^3 + 4x)^{10}$

Clearly: $g \circ f$ is differentiable. (We can use the Binomial Thm to expand & then take the derivative as a polynomial. But this sounds terrible to do! Instead, with Chain rule, this is very easy:

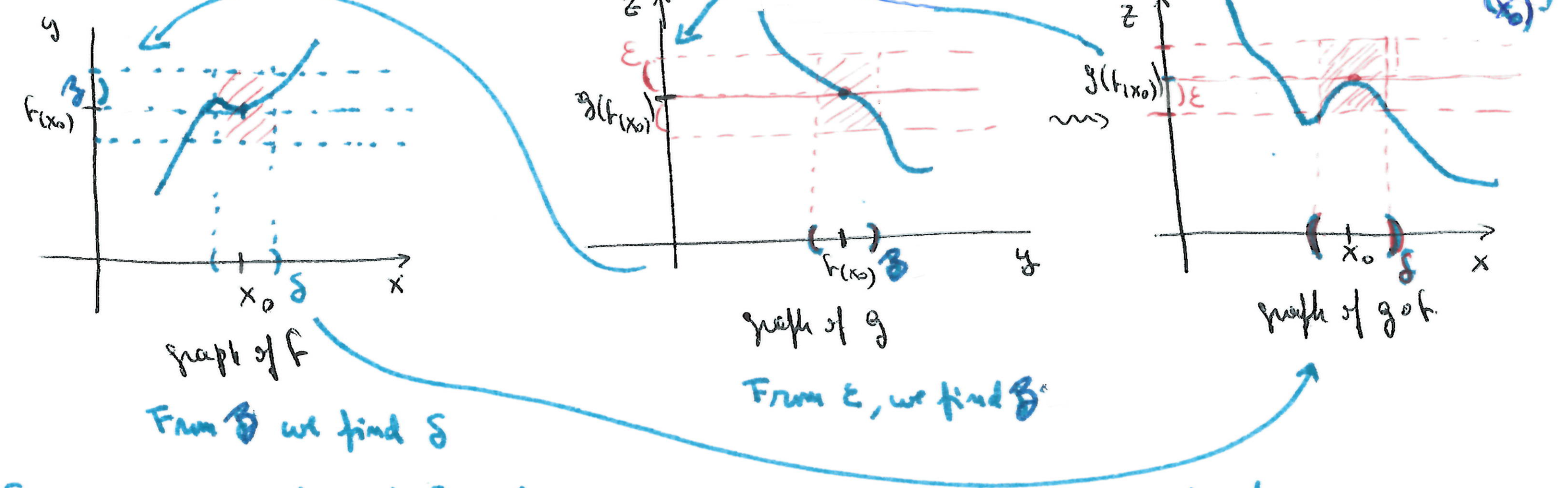
$g'(y) = 10\, y^9 \qquad f'(x) = 3x^2 + 4 \qquad$ so $(g \circ f)' = 10\,(x^3 + 4x)^9 (3x^2 + 4)$

In general $h(x) = f(x)^n$ for $n$ integer gives $h'(x) = n\, f(x)^{n-1} \cdot f'(x)$.

[2]

**Proof of THM 1**: want to show $\lim_{x \to x_0} g\circ f(x) = g\circ f(x_0)$ via $\varepsilon/\delta$.

Given $\varepsilon > 0$ want to find $\delta > 0$ so that if $0 < |x - x_0| < \delta$ then $|g(f(x)) - g(f(x_0))| < \varepsilon$




- Since $g$ is cont. at $y_0 = f(x_0)$, we can find $\delta' > 0$ so that if $0 < |y - y_0| < \delta'$, then $|g(y) - g(y_0)| < \varepsilon$ (*)

→ We would like to ~~plug~~ $y = f(x)$, but for this to happen, we need $|f(x) - f(x_0)| < \delta'$
How can we get this? By continuity of $f$!
For $\tilde{\varepsilon} = \delta' > 0$ we can find $\delta > 0$ so that if $0 < |x - x_0| < \delta$, then $|f(x) - f(x_0)| < \delta'$.

- Follow the choices in reverse order: if $0 < |x - x_0| < \delta$ then $|\underbrace{f(x)}_{=y} - \underbrace{f(x_0)}_{=y_0}| < \delta'$, and so by (*) $|g(f(x)) - g(f(x_0))| < \varepsilon$. □

Message: $\delta'$ was the middle man that allowed us to find $\delta$ from $\varepsilon$.

**Proof of THM 2**: Use the definition!

$$(g\circ f)'(x) = \lim_{\Delta x \to 0} \frac{g(f(x+\Delta x)) - g(f(x))}{\Delta x}$$

want to see $f(x+\Delta x)$ as $f(x) + \Delta y$
$\Delta y := f(x+\Delta x) - f(x)$

$$= \lim_{\Delta x \to 0} \frac{g(f(x) + \Delta y) - g(f(x))}{\Delta x}$$

$$(**) = \lim_{\Delta x \to 0} \frac{g(f(x) + \Delta y) - g(f(x))}{\Delta y} \cdot \frac{\Delta y}{\Delta x}.$$

We have 2 factors to treat & use product rule for limits!

• Since f is continuous at x because it's differentiable, we have $\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} f(x+\Delta x) - f(x) = 0$.

So $$\lim_{\Delta x \to 0} \frac{g(f(x)+\Delta y) - g(f(x))}{\Delta y} \dot{=} g'(f(x))$$ (if $\Delta x \to 0$ then $\Delta y \to 0$ !)

• Now $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x).$$ □

⚠ How do we know $\Delta y \neq 0$ for $\Delta x$ near 0? → Problem in (**)

If not, this means that we can always find $\Delta x$ as close as 0 as we want with $f(x+\Delta x) - f(x)$. In that case,

$$\frac{g(f(x+\Delta x)) - g(f(x))}{\Delta x} = \frac{0}{\Delta x} = 0 \xrightarrow[\Delta x \to 0]{} 0$$ for these values.

But this is no problem because $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$.

And so $(g \circ f)'(x) = 0$ & also $g'(f(x))\, f'(x) = 0$.

<u>Alternative argument</u> (Artin's proof) Avoids dividing by "$\Delta y$" (~~that~~ because it could be 0) & $f(x+\Delta x) - f(x)$

Write $$\varepsilon(\Delta y) = \frac{g(y+\Delta y) - g(y)}{\Delta y} - g'(y) \xrightarrow[\Delta y \to 0]{} 0$$

Equivalently: $$\boxed{g(y+\Delta y) - g(y) = \Delta y\, g'(y) + \Delta y\, \varepsilon(\Delta y) \quad \text{with } \varepsilon(\Delta y) \xrightarrow[\Delta y \to 0]{} 0}$$ (***)

Now: $h(x) = g \circ f(x)$ satisfies

$$\frac{h(x+\Delta x) - h(x)}{\Delta x} = \frac{g(y+\Delta y) - g(y)}{\Delta x}$$ with $y = f(x)$, $\Delta y = f(x+\Delta x) - f(x)$

replace (***)

$$= \frac{\Delta y\, g'(y) + \Delta y\, \varepsilon(\Delta y)}{\Delta x} = \boxed{\frac{\Delta y}{\Delta x} g'(y)} + \frac{\Delta y}{\Delta x} \varepsilon(\Delta y)$$

Since $\Delta y \xrightarrow[\Delta x \to 0]{} 0$ we get $\boxed{\frac{\Delta y}{\Delta x} g'(y)} \xrightarrow[\Delta x \to 0]{} f'(x)\, g'(f(x))$ & $\frac{\Delta y}{\Delta x} \varepsilon(\Delta y) \to f'(x) \cdot 0 = 0$