Lecture X §3.4 Some trigonometric functions

So far, we have derivation rules for • addition, scalar mult, product, quotient

$(f+g)'=f'+g'$ $\quad (cf)'=cf'$ $\quad (fg)'=f'g+fg'$ $\quad \left(\frac{f}{g}\right)'=\frac{-g'}{g^2}$

• composition (Chain rule) $(g\circ f)'_{(x)}=g'(f_{(x)})\cdot f'_{(x)}$

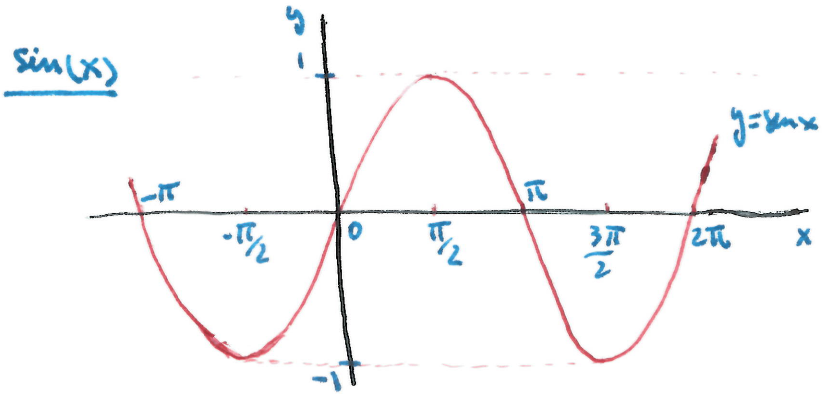
⇝ Derivatives of polynomials, rat'l functions, powers.

Q: What about other functions? Can we have more building blocks than just x^n?

§1 Trigonometric functions: $\sin x$ & $\cos x$.

2 Basic ones: $\sin(x)$ & $\cos(x)$ $\quad \left[\tan(x)=\frac{\sin(x)}{\cos(x)}\right]$

$\sin(x)$



Properties • $\sin(x)$ is PERIODIC with period 2π ($\sin(x+2\pi)=\sin x$)

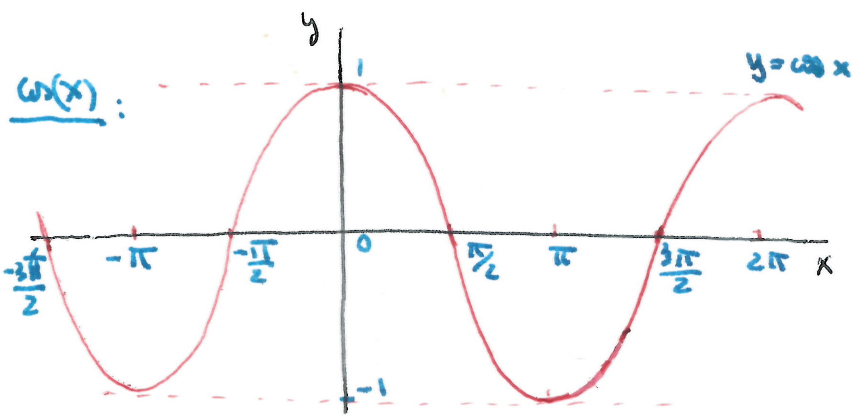
• $\sin(x)$ is ODD ($\sin(-x)=-\sin(x)$ for all x)

• $-1\leq \sin(x)\leq 1$ for all x

Useful Values:

$$\begin{cases} \sin(0)=\sin\pi=0 \\ \sin\frac{\pi}{2}=1, \quad \sin\frac{3\pi}{2}=-1 \\ \sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}, \quad \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2} \quad \sin\frac{\pi}{6}=\frac{1}{2}\end{cases}$$

$\cos(x)$:



Properties • $\cos(x)$ is PERIODIC with period 2π

• $\cos(x)$ is EVEN ($\cos(-x)=\cos(x)$ for all x)

• $-1\leq\cos(x)\leq 1$ for all x

Useful Values

$$\begin{cases} \cos(0)=1, \quad \cos(\pi)=-1 \\ \cos\left(\frac{\pi}{2}\right)=\cos\left(\frac{3\pi}{2}\right)=0 \\ \cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}, \quad \cos\left(\frac{\pi}{3}\right)=\frac{1}{2} \quad \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}\end{cases}$$

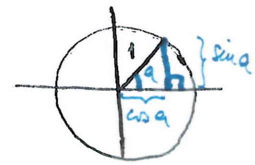
Graph is a shift: $\boxed{\cos(x)=\sin\left(x+\frac{\pi}{2}\right)}$

• These 2 functions vary nice and smoothly, so they should be differentiable. We'll find the formulas by definition, computing the increments. For this, we'll need addition formulas for $\sin(x)$ & $\cos(x)$.

(1) $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$ (mixed)

(2) $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$ (unmixed)

Fundamental eqn (from Pythagoras' Thm) $\boxed{\sin^2 x + \cos^2 x = 1}$



Theorem (1) $\frac{d}{dx}\sin x = \cos x$ & (2) $\frac{d}{dx}\cos x = -\sin x$

Proof Method of increments

(1) $$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} \underset{\substack{\text{Additive}\\ \text{formula 1}}}{=} \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$$

$$\underset{\text{rearrange}}{=} \lim_{\Delta x \to 0} \underbrace{\sin x}_{\substack{\downarrow \Delta x \to 0\\ \sin x}} \boxed{\left(\frac{\cos \Delta x - 1}{\Delta x}\right)} + \underbrace{\frac{\sin \Delta x}{\Delta x}}_{\substack{\downarrow \Delta x \to 0\\ 1}} \underbrace{\cos x}_{\substack{\Delta x \to 0\\ \cos x}}$$

NEEDS MORE WORK! %

Claim $$\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} \frac{\cos \Delta x + 1}{\cos \Delta x + 1} = \lim_{\Delta x \to 0} \frac{\cos^2 \Delta x - 1}{\Delta x(\cos \Delta x + 1)}$$

$$= \lim_{\Delta x \to 0} \frac{\sin^2 \Delta x}{\Delta x(\cos \Delta x + 1)} = \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \frac{\sin \Delta x}{\cos \Delta x + 1} = 1 \cdot 0 = \boxed{0}$$

Conclusion: By limit laws: $\frac{d}{dx}\sin x = \sin x \cdot 0 + 1 \cdot \cos x = \boxed{\cos x}$ ✓

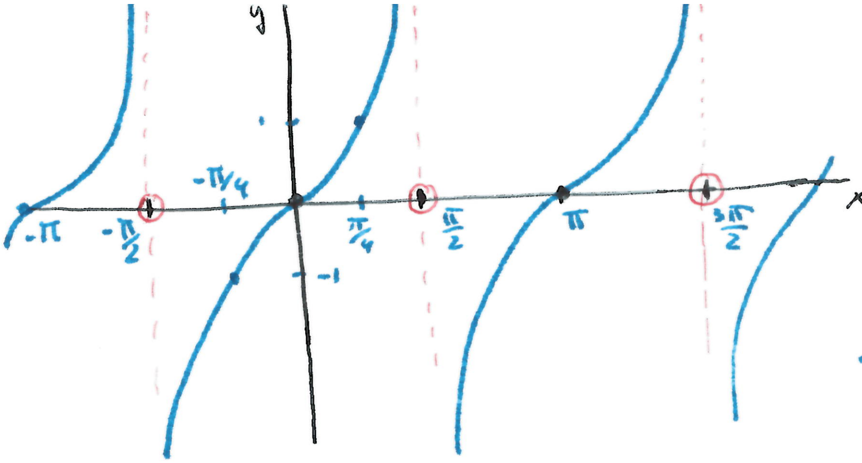
(2) $$\frac{d}{dx}\cos x = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x} \underset{\substack{\text{Additive}\\ \text{formula (2)}}}{=} \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}$$

$$\underset{\text{rearrange}}{=} \lim_{\Delta x \to 0} \underbrace{\cos x}_{\substack{\downarrow \Delta x \to 0\\ \cos x}} \Bigg(\underbrace{\frac{\cos \Delta x - 1}{\Delta x}}_{\substack{\downarrow \Delta x \to 0\\ 0 \text{ (Claim)}}} \Bigg) - \underbrace{\sin x}_{\substack{\downarrow\\ \sin(x)}} \underbrace{\frac{\sin \Delta x}{\Delta x}}_{\substack{\downarrow \Delta x \to 0\\ 1}} \underset{\text{Limit Laws.}}{=} \cos x \cdot 0 - \sin x = \boxed{-\sin x}$$

§2 Other trig functions:

① $\tan x := \frac{\sin x}{\cos x}$ Defined for all x where $\cos x \neq 0$, that is

$$\left.\begin{matrix} x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \\ -\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, \ldots \end{matrix}\right\} \boxed{\begin{matrix} x \neq (2k+1)\frac{\pi}{2} \\ \text{for } k \text{ integer.} \end{matrix}}$$



$$\lim_{x\to\frac{\pi}{2}^-} \frac{\overset{\to 1}{\operatorname{sen} x}}{\underset{\to 0^+}{\cos x}} = +\infty$$

$$\lim_{x\to\frac{\pi}{2}^+} \frac{\overset{\to 1}{\operatorname{sen} x}}{\underset{\to 0^-}{\cos x}} = -\infty$$

__Prop:__
- $\tan(x)$ is PERIODIC with period π
- $\tan(x)$ is ODD $(\tan(-x) = -\tan x)$
- Image of $\tan(x) = \mathbb{R}$.

To get $\frac{d}{dx}\tan x$ we use the Quotient Rule

$$\frac{d}{dx}\tan x = \frac{(\operatorname{sen} x)'\cos x - \operatorname{sen} x\,(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \operatorname{sen}^2 x}{\cos^2 x} = \boxed{\frac{1}{\cos^2 x}} = \sec^2(x)$$

② $\cot(x) = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$. Defined when $\sin x \neq 0$, that is

$x \neq 0, \pi, 2\pi, \ldots, -\pi, -2\pi, \ldots$ $\Big\}$ $\boxed{x \neq k\pi \text{ for } k \text{ integer}}$

$$\frac{d}{dx}\cot(x) = \frac{(\cos x)'\operatorname{sen} x - \cos x\,(\sin x)'}{\operatorname{sen}^2 x} = \frac{-\operatorname{sen}^2 x - \cos^2 x}{\operatorname{sen}^2 x} = \boxed{\frac{-1}{\operatorname{sen}^2 x}} = -\csc^2(x)$$

③ $\sec(x) = \frac{1}{\cos x}$ Domain = Domain $\tan(x)$.

$$\frac{d}{dx}\sec(x) = \frac{-(\cos x)'}{\cos^2 x} = \boxed{\frac{\operatorname{sen} x}{\cos^2 x}} = \tan(x)\sec(x)$$

④ $\csc(x) = \frac{1}{\operatorname{sen} x}$ Domain = Domain $\operatorname{cotan}(x)$

$$\frac{d}{dx}\csc(x) = \frac{-(\operatorname{sen} x)'}{\operatorname{sen}^2 x} = \boxed{\frac{-\cos x}{\operatorname{sen}^2 x}} = -\operatorname{cotan}(x)\csc(x)$$

⚠ We don't need to memorize these formulas, we can just use the Quotient Rule to get them from 2 basic formulas.

__Examples:__

(1) $f(x) = \operatorname{sen}(x^3) \rightsquigarrow g(y) = \operatorname{sen} y$, $h(x) = x^3$ $\quad f = g\circ h \quad g' = \cos y$, $h' = 3x^2$

So $f' = g'(x^3)\,h'(x) = \cos(x^3)\cdot 3x^2$

(2) $f(x) = \operatorname{sen}^3(x)$ $\quad f = h \circ g \quad f' = h'(g(x))\, g' = 3(\operatorname{sen} x)^2 \cos x$

$= 3 \operatorname{sen}^2 x \cos x$

(3) $f(x) = \operatorname{sen}^2 x \cos^3 x$

$$f'(x) = (\operatorname{sen}^2 x)' \cos^3 x + (\operatorname{sen}^2 x)(\cos^3 x)'$$

$$= 2 \operatorname{sen} x \cos x \cos^3 x + \operatorname{sen}^2 x \; 3\cos^2 x \, (-\operatorname{sen} x)$$

$$= 2 \operatorname{sen} x \cos^4 x - 3 \operatorname{sen}^3 x \cos^2 x$$

$$= \cos^2 x \operatorname{sen} x \left(2\cos^2 x - 3 \operatorname{sen}^2 x \right)$$

$$= \cos^2 x \operatorname{sen} x \left(2(\cos^2 x + \operatorname{sen}^2 x) - 5 \operatorname{sen}^2 x \right)$$

$$= \cos^2 x \operatorname{sen} x \left(2 - 5 \operatorname{sen}^2 x \right)$$

(4) $f(x) = \cos((4x+1)^2)$

$$f'(x) = -\sin((4x+1)^2)\; 2(4x+1) \cdot 4 = -8(4x+1) \operatorname{sen}((4x+1)^2)$$