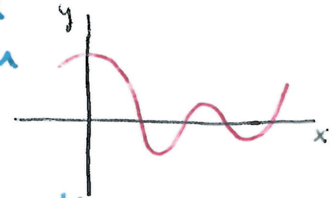
# Lecture XI : §3.5 Implicit functions & fractional exponents

- So far, our functions where given as $f: D \longrightarrow \mathbb{R}$ via a formula

Eg: $y = f(x) = (x^3+4x)^{10}$ or $y = \sin(x)$ $\begin{cases} y = \text{dep. var} \\ x = \text{indep var} \end{cases}$

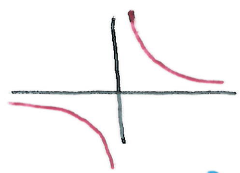



From this we get a curve in the plane = the graph of $f$.

- Often times, we deal with curves given by a relation between the independent variable $x$ & the dependent variable $y$ & we can't solve for $y = f(x)$.

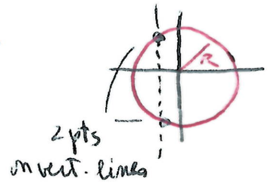
§0 Examples Classical plane curves

① Hyperbola: $xy = 1$



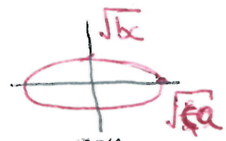
graph of $y = \frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$

② Circle of radius $R$ centered at $(0,0)$ $\quad x^2+y^2=R^2$




NOT the graph of a function because the Vertical Test fails

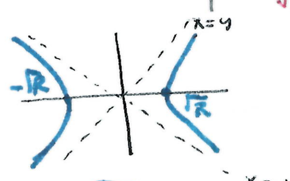
③ Ellipse: $\frac{x^2}{a} + \frac{y^2}{b} = c \quad a, b, c > 0$




$(a>b)$

④. $x^2 - y^2 = R$ is again a hyperbola

$(x-y)(x+y) = R$




$R>0$

Note: For $x^2+y^2=R^2$ we have 2 possible solutions (the half circles above & below x-axis)

$y = \pm\sqrt{R^2-x^2}$

⑤. Equation $2y^2 - 2xy = 10 - x^2 \rightsquigarrow 2y^2 - 2xy + (x^2-10) = 0$

Solve for $y$ with quadratic formula

$$y = \frac{2x \pm \sqrt{4x^2 - 4.2(x^2-10)}}{2.2} = \frac{2x \pm \sqrt{80-4x^2}}{4} = \frac{x \pm \sqrt{20-x^2}}{2}$$

Again, the curve is the union of two graphs $f_+, f_- : [-\sqrt{20}, \sqrt{20}] \longrightarrow \mathbb{R}$

Equations like this one can be "solve by radicals" only if degree in $y$ is $\leq 4$ [Galois Theory]

Locally = curve is the graph of a function unless the Tang line is vertical

GOAL: Want to find slope of the Tangent line to the curve at $(x,y)$.

Q: Can we compute $\frac{dy}{dx}$ locally without the explicit formula, but just using the implicit equation relating $y$ & $x$?

§1 A: Implicit differentiation!

Guiding Principles P1. "If 2 functions are =, so are their derivatives"

P2. Chain rule & various techniques for computing derivatives

P3: Check operations to find bad pts where the formula can fail.

Back to examples:

⑤ $2y^2 - 2xy = 10 - x^2$ Think $y = y(x)$ & take $\frac{d}{dx}$ on both sides using chain rule & product rule (P2). [P1 says 2 derivatives agree!]

$$2(y(x))^2 - 2\,x\,y_{(x)} = 10 - x^2$$

$$\frac{d}{dx}:\ 2\cdot 2\, y_{(x)}\, y' - 2(y + xy') = -2x$$

$$4y\,\boxed{y'} - 2y - 2x\,\boxed{y'} = -2x \quad \rightsquigarrow \text{ Want to solve for } y'$$

so $(4y - 2x)\,y' = 2y - 2x$

$$(2y - x)\,y' = y - x \quad \rightsquigarrow \quad \boxed{y' = \frac{y - x}{2y - x}} \quad \text{if } 2y - x \neq 0$$

Need to check if $2y - x \neq 0$ for points satisfying the origin equation.

How? Set $y = \frac{x}{2}$ & substitute $2\left(\frac{x}{2}\right)^2 - 2x\left(\frac{x}{2}\right) = 10 - x^2$

$$\frac{x^2}{2} - x^2 = 10 - x^2$$

So these 2 pts are problematic! $x^2 = 20$ so $\boxed{x = \pm\sqrt{20}}$

Conclusion: $y' = \frac{y - x}{4y - x}$ as long as $x \neq \pm\sqrt{20}$

(2 pts $\left(\pm\sqrt{20}, \frac{\pm\sqrt{20}}{2}\right)$ must be removed from the curve)

Sanity check: Verify that this is true for our 2 functions $f_+$, $f_-$ that solved the equation.

- For $f_+$: $\quad \frac{df_+}{dx} = \frac{1}{2} + \frac{1}{2}\frac{d}{dx}\sqrt{20-x^2} \underset{\text{Later}}{=} \frac{1}{2} + \frac{1}{2}\,\frac{-2x}{\sqrt{20-x^2}}$.
  $y=f_+(x)$

  But $\sqrt{20-x^2} = 2y-x$ by def. of $f_+$, so $f'_+ = \frac{1}{2} - \frac{x}{2(2y-x)} = \frac{2y-x-x}{2(2y-x)} = \boxed{\frac{y-x}{2y-x}}$

  We got the same answer!
- Similar method works for $f_-$.

(2) $\quad x^2+y^2=R^2 \rightsquigarrow y=y(x)$ gives $x^2+y(x)^2=R^2$

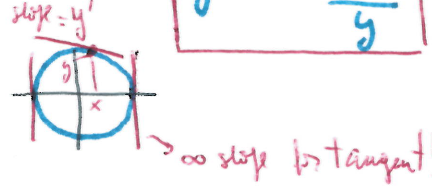
Take $\frac{d}{dx}$ on both sides: $\quad 2x+2y(x)\,y'=0$

$$2y\,y'=-2x$$

$$\boxed{y'=\frac{-x}{y}} \quad \text{as long as } y\neq 0$$

If $y=0$ then $x=\pm\sqrt{R^2}$

- We get the formula everywhere except at these 2 pts.




Double check for $y=\pm\sqrt{R^2-x^2} \rightsquigarrow y' = \pm\frac{1}{2}\frac{(-2x)}{\sqrt{R^2-x^2}} = -\frac{x}{\pm\sqrt{R^2-x^2}} = \frac{-x}{y}$

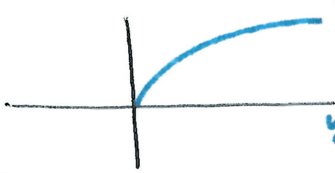
- Tang line through $\left(\frac{R}{2}, \frac{\sqrt{3}}{2}R\right)$? slope: $y' = \frac{-R/2}{\sqrt{3}/2\,R} = \frac{-1}{\sqrt{3}} \rightsquigarrow$ Eqn $\boxed{y=\frac{-1}{\sqrt{3}}\left(x-\frac{R}{2}\right)+\frac{\sqrt{3}}{2}R}$

## §2 Applications:

**App (1)**: Derivative of fractional exponents $\quad y = x^{p/q}$ $\quad p, q$ integers, $q\neq 0$.

Meaning $\quad y^q = x^p \qquad$ **CLAIM** $\quad y' = \frac{p}{q}x^{\frac{p}{q}-1}$

Eg $\frac{p}{q}=\frac{1}{2}$:



$y=\sqrt{x}=x^{1/2}$
$[x=y^2]$
$y'=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

Think $y=y(x)$ & use impl. diff. on $y_{(x)}^q = x^p$ via power rule!

$$q\,y^{q-1}(x)\,y'(x) = p\,x^{p-1} \quad\text{so}\quad y' = \frac{p}{q}\,\frac{x^{p-1}}{y^{q-1}}$$

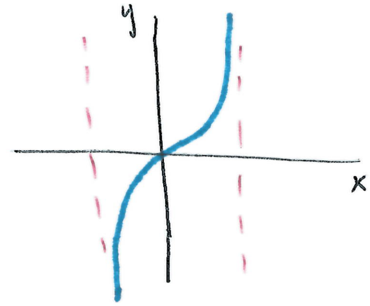
But $y^q = x^p$ so $y^{q-1} = \frac{x^p}{y} = \frac{x^p}{x^{p/q}} = x^{p-p/q}$

Then $y' = \frac{p}{q}\,\frac{x^{p-1}}{x^{p-p/q}} = \boxed{\frac{p}{q}\,x^{\frac{p}{q}-1}} \rightsquigarrow$ same Power Rule works with fractional Exponents!

**Eg 2**: $\quad y=\sqrt{\cos x} \underset{\text{Chain Rule}}{\rightsquigarrow} y' = \frac{1}{2}\frac{1}{\sqrt{\cos x}}(\cos x)' = \frac{-\operatorname{sen} x}{2\sqrt{\cos x}}$.

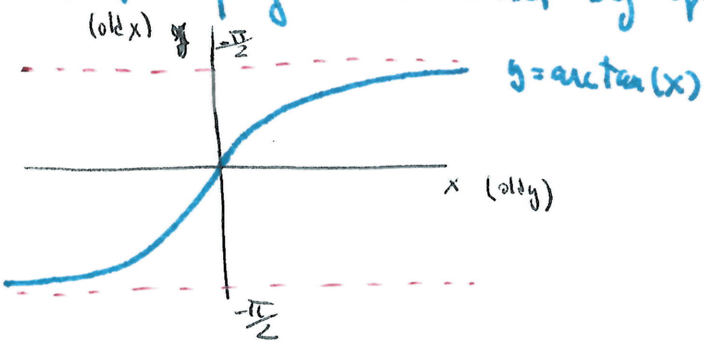
App ② Derivative of inverse trig functions

Eg $\tan(x) = y$ $\quad \tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$




It has an inverse function $g = g(y)$ called <u>arc tan</u> : meaning $\begin{cases} g(\tan(x)) = x \\ \tan(g(x)) = x \end{cases}$

The graph of $g$ is obtained by flipping the graph & the axes.




- Want to find $y'$ <u>only in terms of</u> $x$
- Use $x = \tan(y)$ equ. & impl. diff. Think $y = y(x)$.

- $x = \tan(y(x)) \xrightarrow[\frac{d}{dx}]{} 1 = y'(\tan y)' = \frac{y'}{\cos^2 y}$ so $y' = \cos^2 y$

Still not good enough!

- Can go further $x^2 = \tan^2(y) = \frac{\sin^2 y}{\cos^2 y} = \frac{1 - \cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y} - 1$

So $\frac{1}{\cos^2 y} = x^2 + 1$ gives $\cos^2 y = \frac{1}{x^2+1}$

We get $\boxed{\frac{dy}{dx} = \frac{1}{x^2+1}}$ for $y = \arctan(x)$.