Lecture XII §3.6 Derivatives of Higher Order
Appendix A4: The Mean Value Theorem

§1 Derivatives of higher order

Simple idea: If $f: D \longrightarrow \mathbb{R}$ is differentiable, then $f': D' \longrightarrow \mathbb{R}$, $x \longmapsto f'(x)$ is a function defined on a (possible smaller) set D'. If f' is a differentiable function, we can differentiate again, and write $f'' := (f')': D'' \longrightarrow \mathbb{R}$, $x \longmapsto (f')'(x)$, and so on...

Notation: y'', $f''(x)$, $f^{(2)}(x)$, $\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2}$

y''', $f'''(x)$, $f^{(3)}(x)$, $\frac{d^3f}{dx^3}$

$\vdots$

In general: $y^{(n)}$, $f^{(n)}(x)$, $\frac{d^nf}{dx^n}$ for $n \geq 1$ integer

Convention: $f^{(0)}(x)$ means $f(x)$.
(see Example 3 next page)

Example 1: Monomials ($\leadsto$ polinomials by additivity)

$y = x^n$ $n > 0$ integer or $y = c$ constant $\hookrightarrow c' = 0,\ c'' = 0, \ldots, c^{(m)} = 0$ for all $m > 0$

$\downarrow$

$y'(x) = n x^{n-1}$, $y''(x) = n(n-1)x^{n-2}$, $y^{(3)}(x) = n(n-1)(n-2)x^{n-3}, \ldots$

When does this stop? $y^{(k)} = n(n-1)\ldots(n-k+1)\, x^{n-k}$ for $k \leq n$

$y^{(n+1)} = 0$ so $y^{(k)} = 0$ for $k > n$

Note: We can shorten the notation with $\begin{cases} p! = p(p-1)(p-2)\cdots 2\cdot 1 & \text{for } p \geq 1 \text{ integer} \\ 0! = 1 \end{cases}$ "p factorial"

In this case $y^{(k)} = n(n-1)\cdots(n-k+1)\dfrac{(n-k)(n-k-1)\cdots 1}{(n-k)(n-k-1)\cdots 1}\, x^{n-k}$

$$\boxed{y^{(k)}(x) = \begin{cases} \dfrac{n!}{(n-k)!}\, x^{n-k}. & \text{for } k \leq n \\ 0 & \text{for } k > n \end{cases}}$$

Example 2: Monomials with negative powers

$y = x^{-n} = \frac{1}{x^n}$ for $n > 0$ integer $\leadsto y' = \frac{-n}{x^{n+1}}$, $y'' = \frac{n(n+1)}{x^{n+2}}$,

$$y''' = \frac{-n(n+1)(n+2)}{x^{n+3}}, \ldots$$ so it never ends!

In general: $$y^{(k)} = (-1)^k \frac{n(n+1)\cdots(n+k-1)}{x^{n+k}}$$ for all $k>0$ integer

$$= \frac{(-1)^k (n+k-1)!}{x^{n+k}(n-1)!}$$

<u>Example 3</u> Trig functions

- $y = \sin x$, $y' = \cos x$, $y'' = -\sin x$, $y''' = -\cos x$, $y^{(4)} = \sin x$

& it repeats from here. Same happens to $y = \cos x$.

<u>Remark</u>: $\sin(x)$ & $\cos(x)$ will <u>both</u> satisfy $y'' = -y$.

In fact, the solutions to this $\underset{y''+y=0}{\text{ODE}}$ are always of the form

$y(x) = a\sin(x) + b\cos(x)$ for a, b in $\mathbb{R}$ ("Linear combinations of sin & cos")

⇝ Simple Harmonic motion (§ 9.6)

<u>Obs</u>: We can <u>combine</u> higher derivatives with implicit differentiation

<u>Eg</u> $x^2+y^2=R^2$ & think $y=y(x)$. By impl diff, we get

(*) $\boxed{2x+2yy'=0}$ ⇝ $y' = -\frac{x}{y}$ if $y \neq 0$.

y' is diff'ble, so look at (*) & differentiate again:

$$2 + 2(y'y' + yy'') = 0$$ use $y' = \frac{-x}{y}$

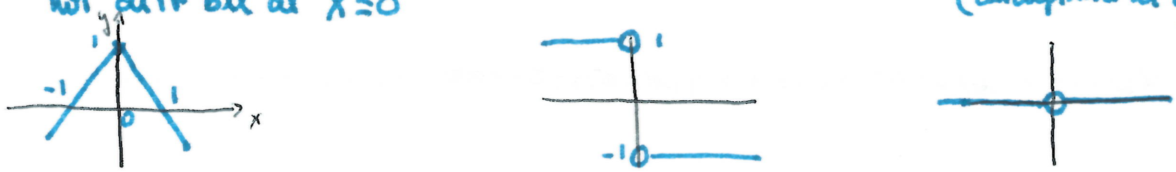
$$2 + 2\left(\left(\frac{-x}{y}\right)^2 + y\,y''\right) = 0$$

So $1 + \frac{x^2}{y^2} + y\,y'' = 0$ gives: $y\,y'' = \frac{-(y^2+x^2)}{y^2} = \frac{-R^2}{y^2}$

$$\boxed{y'' = \frac{-R^2}{y^3}}$$ for $y \neq 0$. (2 pts removed)

<u>Example 3</u> $f(x) = 1-|x| = \begin{cases} 1-x & x \geq 0 \\ 1+x & x<0 \end{cases}$ ⇝ $f'(x) = \begin{cases} -1 & x>0 \\ 1 & x<0 \end{cases}$ ⇝ $f''(x) = 0$ for $x \neq 0$ (undefined at 0)

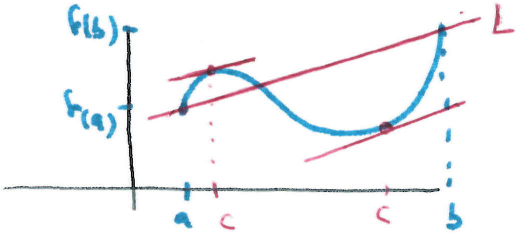
not diff'ble at $x=0$



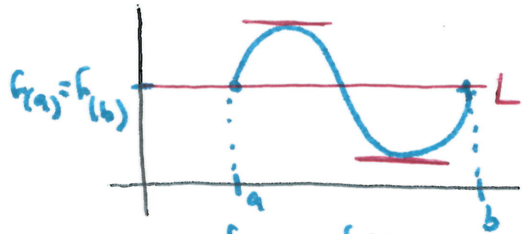
§2 Appendix A4: Mean Value Theorem

Thm (MVT): If $f:[a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ & diff'ble on (a,b), there exists c in (a,b) with $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

↳ slope of line through $(a, f(a))$ & $(b, f(b))$



$f(a) \neq f(b)$
General case



$f(a) = f(b)$
Special case: ROLLE'S THM.

We'll see that from a "different perspective" we only need to check the special case:

ROLLE'S THM: $f:[a,b] \longrightarrow \mathbb{R}$ cont on $[a,b]$ & diff'ble on (a,b), AND $f(a) = f(b)$, then we can find c in (a,b) with $f'(c) = 0$

Proof Recall that by the Extreme Value Thm (EVT) we know f has a max & min value. Unless f is constant (in which case $f' = 0$ always & any c works) we have that the max & min values are different. In particular, since $f(x) \neq f(a)$ we can find x with $f(x) > f(a)$ or $f(x) < f(a)$. Start with the case $f(x) > f(a)$ [the other case has exactly the same proof] So $f(a)$ is not the max & so we have that the max is achieved at some c in (a,b). Since f is differentiable, we must have $f'(c) = 0$. □

Note: Key was to show either the max or the min happen in (a,b). & then use c extremal & f differentiable at c forces $f'(c) = 0$.

• From Rolle to MVT.

Proof: We write an auxiliary function that will satisfy Rolle's conditions

Recall Eqn of the line L through $(a, f(a))$ & $(b, f(b))$ is

$$y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a) \qquad \text{or } 0 = y - \left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)$$

We build the function $g:[a,b] \longrightarrow \mathbb{R}$ $g(x) = f(x) - \left(\dfrac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)$

Properties of g:
- g is continuous on $[a,b]$
- $g(x)$ is differentiable on (a,b)
- $g(a) = g(b) = 0$

So by ROLLE'S THM applied to $g(x)$, we can find c in (a,b) with $g'(c)=0$.

But $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \cdot 1$ so $g'(c)=0$ gives $f'(c) = \frac{f(b)-f(a)}{b-a}$ $\square$

- What does $g(x)$ do? It tests if $(x, f(x))$ lies on the line L or not ($g(x)=0$ means yes, $g(x)\neq 0$ — NO)

- We can generalize MVT.

Thm (General MVT) Pick $f,g:[a,b] \longrightarrow \mathbb{R}$ continuous on $[a,b]$ & differentiable on (a,b). Assume $g'(x) \neq 0$ for all x in (a,b). Then we can find c in (a,b) with $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Note: $g(x)=x$ gives back MVT.

Proof: First, we argue $g(b) \neq g(a)$. If not, by Rolle's Thm applied to g, we can find $\tilde{c}$ in (a,b) with $g'(\tilde{c})=0$, but we know this can't happen for g by our assumptions.

As before, we build an auxiliary function, apply ROLLE to it (so we find a c) and conclude our formula is true for that value c.

Set $h:[a,b] \longrightarrow \mathbb{R}$ $\quad h(x) = (f(b)-f(a))(g(\underline{x})-g(a)) - (f(\underline{x})-f(a))(g(b)-g(a))$

(comes from doing the cross product in the formula we want to prove).

Properties of h:
- h is cont on $[a,b]$
- h is diff'ble on (a,b)
- $h(a) = h(b) = 0$

By ROLLE'S THM applied to $h(x)$, we can find c in (a,b) with $h'(c)=0$.

But $h'(x) = (f(b)-f(a))\, g'(x) - f'(x)\,(g(b)-g(a))$

So $0 = h'(c) = f(b)-f(a)\; g'(c) - f'(c)\,(g(b)-g(a))$ gives

$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$ as we wanted.