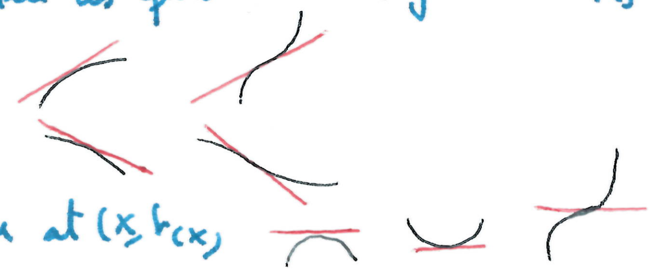
§1 Max & Min Problems. Growth of functions

GOAL: Use f' (& higher order derivatives) to sketch the graph of $f(x)$.

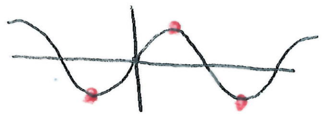
Q: Assume f is differentiable. What does f' tell us quantitatively about $f(x)$?

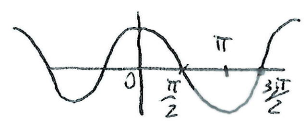
- $f'(x) > 0 \rightsquigarrow f$ is str. increasing
- $f'(x) < 0 \rightsquigarrow$ ―――――― decreasing
- $f'(x) = 0 \rightsquigarrow f$ has a horiz tangent line at $(x, f(x))$

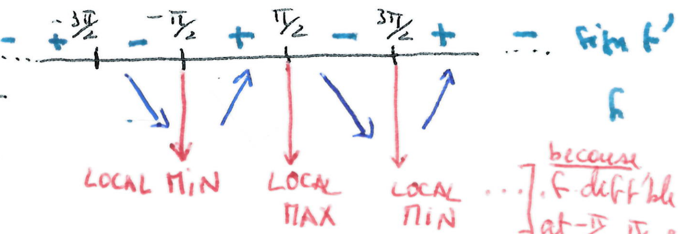


So the sign of f' determines a lot $\rightsquigarrow$ Look for regions where the sign of f' is constant.

Example 1 $f(x) = \sin(x) \rightsquigarrow f'(x) = \cos x$



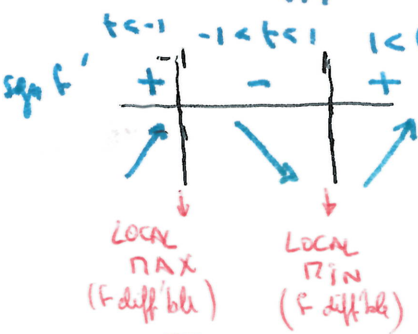




... because f diff'ble at $-\frac{\pi}{2}, \frac{\pi}{2}$ & $\frac{3\pi}{2}$

Example 2 $f(t) = t^5 - 5t + 1 \rightsquigarrow f'(t) = 5t^4 - 5 = 5(t^4 - 1) = 5(t^2+1)(t^2-1)$

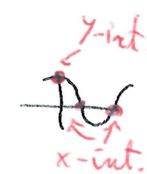
$= 5\underbrace{(t^2+1)}_{>0 \text{ always}}(t-1)(t+1)$



Also: $\lim\limits_{t\to\infty} f(t) = \infty$, $\lim\limits_{t\to-\infty} f(t) = -\infty$ } so f has NO extreme values in $\mathbb{R}$

Useful Things to determine the graph of f:

1. Critical points = either $f'(x) = 0$ or $f(x)$ is undefined (eg $f(x) = |x|$ has 0 as a crit pt)
2. Critical Values = $f(x)$ for x a critical pt
3. Sign of $f'(x)$ between crit. pts & between points where f is not defined (eg $f(x) = \frac{1}{x(x+1)}$ $x = 0, -1$)
4. Intercepts: $f(0)$ (y-intercept) & $f(x) = 0$ (x-intercept) CAN BE HARD (EX 2)



5. $\lim\limits_{x\to\infty} f(x)$ & $\lim\limits_{x\to-\infty} f(x) \rightsquigarrow$ Potential asymptotes, $\pm\infty$, or no limit
6. Behavior of f near the points where f is not defined (eg $\lim\limits_{x\to0^-}$ & $\lim\limits_{x\to0^+}$ for $f = \frac{1}{x(x+1)}$)
7. Parity / Periodicity (only if just trig functions appear)

<u>Back to Ex 2</u>: x-intercepts are hard to find. We can ONLY guess the location & number of zeros using Interm. Value Theorem (IVT)

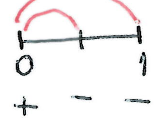
<u>How?</u> • $f(0)=1$, $f(1)=-3$ f cont $\rightsquigarrow$ we have a zero in $(0,1)$.

• $\left.\begin{array}{l} f(-1)=5>0 \\ f(-2)=-21<0 \end{array}\right\} \underset{IVT}{\rightsquigarrow}$ we have a zero in $(-2,-1)$.

• $\left.\begin{array}{l} f(1)=-3 \\ f(2)=23>0 \end{array}\right\} \underset{IVT}{\rightsquigarrow}$ ——————— $(1,2)$.

Growth pattern of f says we can't have more than 3 zeros.

• Numerically, we can cut each segment in half to improve our search. (and repeat!)

Eg: $f(\frac{1}{2}) = -\frac{47}{32} < 0$  $\Rightarrow$ zero in $(0,\frac{1}{2})$.

<u>Example 3</u> $f(x) = \sec(x) = \dfrac{1}{\cos(x)}$ $\rightsquigarrow$ 7. $\begin{cases} \text{• Periodic (with period } 2\pi) \\ \text{• } \dfrac{1}{\cos(-x)} = \dfrac{1}{\cos(x)} \text{ so EVEN function} \end{cases}$

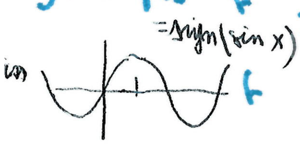
$f'(x) = \sec(x)\tan(x) = \dfrac{\sin(x)}{\cos^2(x)}$ undef at $\cos(x)=0$ (but f is also undefined there)

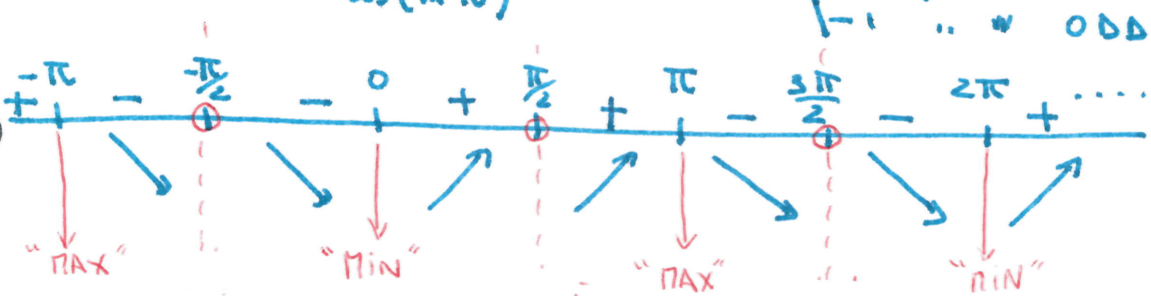
1. <u>Crit pts</u>: • $x = \ldots -\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ $\longleftarrow$ ($x = (2k+1)\frac{\pi}{2}$ for k integer.)

• $f'(x)=0$ if $\sin(x)=0$ $\rightsquigarrow$ $\left.\begin{array}{l} 0, \pi, 2\pi, \ldots \\ -\pi, -2\pi, \ldots \end{array}\right\}$ $x = m\pi$ m integer

2. <u>Crit Values</u>: $f(m\pi) = \dfrac{1}{\cos(m\pi)} = \pm 1$ $\left(\begin{array}{l} +1 \text{ for } m \text{ EVEN} \\ -1 \text{ '' '' ODD} \end{array}\right)$

3. Signs f' $= \text{sign}(\sin x)$



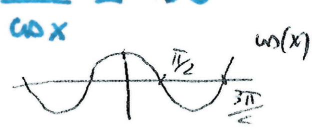


4. Zeros of f: NONE $\rightarrow$ no x-intercepts, $f(0) = \dfrac{1}{\cos(0)} = 1$ y-int.

5. Behavior at $\pm\infty$: no limit exists because f is <u>periodic</u>

6. $\displaystyle\lim_{x\to\pi/2^-} f(x) = \lim_{x\to\pi/2^-} \frac{1}{\cos x} = +\infty$ & $\displaystyle\lim_{x\to\pi/2^+} f(x) = \lim_{x\to\pi/2^+} \frac{1}{\cos x} = -\infty$

$\displaystyle\lim_{x\to(3\pi/2)^-} \frac{1}{\cos(x)} = +\infty$, $\displaystyle\lim_{x\to(3\pi/2)^+} \frac{1}{\cos x} = -\infty$



Graph of $\frac{1}{\cos(x)}$: