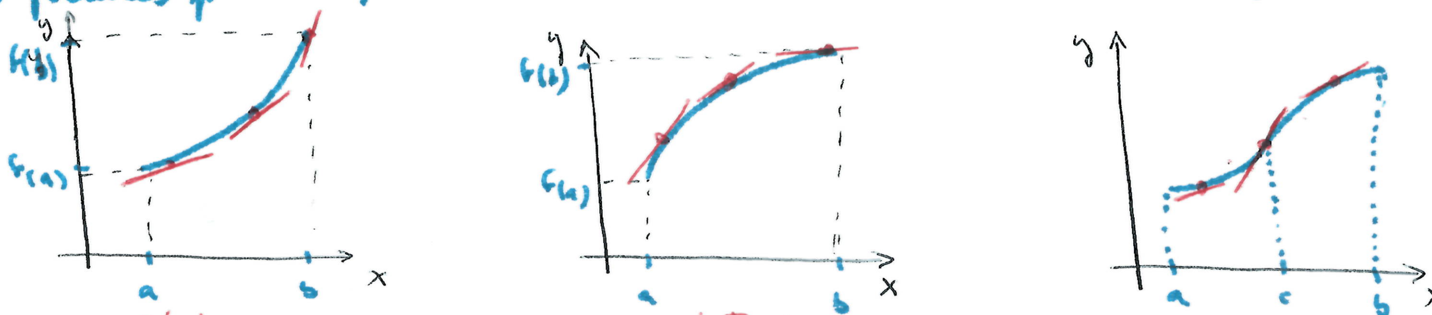
Lecture XIV: §4.2 Concavity & points of inflection

Last time: Used f' to study growth of f, local extrema & extreme values

TODAY: Use higher order derivatives to study convexity = "bending" of the graph of f.

Key fact: $f'' = (f')'$ so f'' gives us growth info on f' = slope of tangent lines to f!

3 pictures for $f: [a,b] \longrightarrow \mathbb{R}$

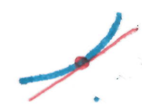


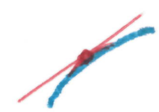
f' str. incr. — CONCAVE UP(WARDS)

f' str. decr — CONCAVE DOWN(WARDS)

f' str incr on (a,c)
f' — decr on (c,b)
c = point of inflection

· **Formally**:

Def: If the graph of f lies ABOVE all of its tangent lines on the interval $[a,b]$, we say f is concave up(wards) on $[a,b]$ 

· If the graph of f lies BELOW all of its tangent lines ______ $[a,b]$, we say f is concave down(wards) on $[a,b]$ 

· **Q**: How to test this without drawing?

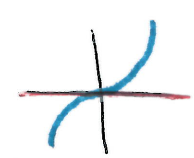
Concavity Test: Assume f' is differentiable on (a,b)

(1) If $f'' > 0$ on (a,b) then f is concave UP on (a,b) (write C.U.)

(2) If $f'' < 0$ ______ DOWN ______ (— C.D.)

Why? On (1) f' is increasing & on (2) f' is decreasing. (see page 4)

Example 1 $f(x) = x^3$. Find intervals when f is CU/CD.



$f'(x) = 3x^2$, $f''(x) = 6x$ ⟿ $f''(x) = 0$ for $x = 0$

		0	
sign f''	$-$		$+$
f	CD		CU

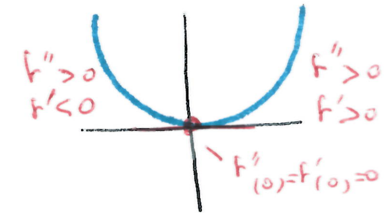
At $x=0$ change of concavity & graph of f lies on both sides of tang line $y=0$.

Def A point c in the domain of f is an <u>inflection point</u> if f is continuous at c & the function changes concavity at c

<u>Remark</u>: Inflection points satisfy: $f''(x) = 0$ or f'' is <u>NOT</u> defined at x [critical pts of f']

Same properties are true for local max/min of f, so they are <u>NOT sufficient</u> to characterize inflection pts.

<u>Example</u> 2 $f(x) = x^4$ has a local minimum at $x = 0$



$f'(x) = 4x^3$, $f'' = 12x^2$ so $f''(x) = 0$ gives $x = 0$

sign f'' $\quad$ + $\;$ 0 $\;$ + $\quad$ CU | CU $\quad\Rightarrow$ $x = 0$ is <u>NOT</u> an inflection pt even though $f''(0) = 0$

This example leads to the following criterion to find local max/min

<u>The Second Derivative Test</u>: Suppose $f''(x)$ is continuous near $x = c$

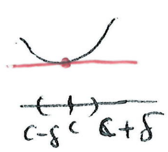
(1) If $f'(c) = 0$ & $f''(c) > 0$, then f has a local min at c

(2) ———————— $f''(c) < 0$ ———————————— max at c

Why? . $f'(c) = 0$ says that tangent line is horizontal.

(1) If $f''(c) > 0$, but cont of f'', then we can find $\delta > 0$ with $f''(x) > 0$ for $c - \delta < x < c + \delta$

So f'' is CU in the interval $(c-\delta, c+\delta)$, ie the graph is above the tangent line $y = f(c)$. so $f(x) > f(c)$ in this interval & c gives a local MIN of f.



(2) Same arguments work for (2).

⚠ This test <u>CANNOT</u> be used if $f''(c) = 0$. It also fails when $f''(c)$ does not exist. (eg $f(x) = x^3$ $x = 0$ is infl pt, no max, no min $f'(0) = f''(0) = 0$)

<u>Examples</u>: ① Find the local max/min & inflection pts of
$$f(x) = 1 + 3x^2 - 2x^3.$$

Soln : . f is diff'ble everywhere & up to any order, so local max/min are crit pts with $f'_{(x)}=0$.

- $f'(x) = 6x - 6x^2 = 6x(1-x)$ ⇝ zeroes : $x=0$, $x=1$.

  $f''(x) = 6 - 12x = 6(1-2x)$ ⇝ — : $x = 1/2$

We draw a table with signs of f', f''

	$x<0$	$0<x<\frac{1}{2}$	$\frac{1}{2}<x<1$	$x>1$
signs f''	+	+	−	−
" f'	$-+=\ominus$	$++=\oplus$	$\oplus$	$\ominus$
f	CU dec	CU inc	CD inc	CD dec

At $x=0$: MIN; at $x=\frac{1}{2}$: infl pt; at $x=1$: MAX

f'' cont. We use the 2nd deriv Test

$$\begin{cases} \cdot\ f'(0)=0 \ \& \ f''(0)>0 \Rightarrow 0 \text{ is the MIN} \\ \cdot\ f'(1)=0 \ \& \ f''(0)<0 \Rightarrow 1 \text{ is the MAX} \end{cases}$$

Alternatively :

$$\left.\begin{array}{lll} f'(x)<0 & \text{for } x<0 & \text{near } 0 \\ f'(x)>0 & \text{" } x>0 & \text{" } 0 \end{array}\right\} \Rightarrow x=0 \text{ is local MIN.}$$

$$\left.\begin{array}{lll} f'(x)>0 & \text{" } x<1 & \text{near } 1 \\ f'(x)<0 & \text{" } x>1 & \text{" } 1 \end{array}\right\} \Rightarrow x=1 \text{ " " MAX.}$$

Change in concavity at $x=\frac{1}{2}$ ⇝ only an inflection pt !

② $f(x) = \sin(x)$ ⇝ $f'(x) = \cos(x)$ ⇝ $f''(x) = -\sin(x)$ cont.

- $f''(x) = 0$ for $x = 0, \pi, 2\pi, \dots$
  $-\pi, -2\pi, \dots$

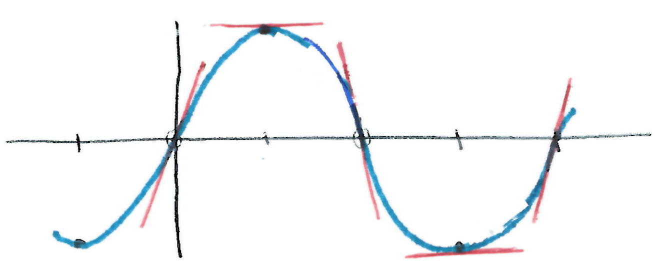
- $f'(x) = 0$ for $x = \pi/2, \frac{3\pi}{2}, \dots$
  $-\pi/2, -\frac{3\pi}{2}, \dots$

	$x<0$	$0<x<\frac{\pi}{2}$	$\frac{\pi}{2}<x<\pi$	$\pi<x<\frac{3\pi}{2}$	$\frac{3\pi}{2}<x<2\pi$	$x>2\pi$
sign f''	+	−	−	+	+	−
" f'	+	+	−	−	+	+
f	CU INC	CD INC	CD DEC	CU DEC	CU INC	CD INC

At 0: infl pt; at $\frac{\pi}{2}$: L MAX; at π: infl pt; at $\frac{3\pi}{2}$: L MIN; at 2π: infl pt.

2nd Derivative Test :

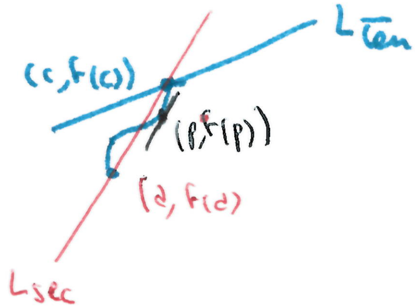
$$\begin{cases} f'(\pi/2) = 0 & f''(\pi/2) < 0 \Rightarrow x = \pi/2 \text{ is local MAX} \\ f'(\frac{3\pi}{2}) = 0 & f''(\frac{3\pi}{2}) > 0 \Rightarrow x = \frac{3\pi}{2} \text{ " " MIN.} \end{cases}$$



Exercise : Do the same analysis for $f(t) = t^5 - 5t + 1$ (last time)

Proof of Concavity Test: We prove only (1), since (2) is very similar.

We know $f''(x) > 0$ on (a,b) & we want to show that the graph of f lies ABOVE the tangent line at $(c, f(c))$ for any $a<c<b$. We argue by contradiction & assume it fails for some c. This means that we can find a point d as close to c as desired with $f(d)$ below the tang line (say $d<c$)



Since f'' exists, f' is diff'ble & so it's continuous.
In particular: f is diff'ble everywhere, so on $[d,c]$
f " continuous on (d,c)

By the Mean Value Thm, we can find a point p in (d,c) with $f'(p) = \dfrac{f(c)-f(d)}{c-d}$ = slope of secant > slope of Tangent at $(c,f(c))$ $= f'(c)$

So $f'(p) > f'(c)$ with $p<c$. This means f' is not increasing, contradicting $f''>0$ on (a,b). (*)

We conclude no such point d can exist so f is CU near any c.

(*) [If $d>c$ we get $f'(p) < f'(c)$ & $p>c$ so f' will not be increasing, which is again a contradiction.]