Lecture XV: §4.3 Applied Maximum & Minimum Problems
§4.4 More max-min problems.

TODAY'S MESSAGE: The solution of many applied problems revolves around maximizing or minimizing something

CHALLENGE: Translation to a concrete function to maximize / minimize

Examples ① In Physical sciences, nature often wants to minimize something

(1) energy used (Hamilton's Principle of LEAST ACTION)

(2) Time to travel from A to B (Fermat's Principle of LEAST TIME)

so often the solutions come from a MINIMIZATION process.

② In Business, the simplest paradigm is:

(1) minimize costs

(2) maximize profit.

KEY STEPS:
- Modeling = Find the equations that govern the problem.
- Determine the constraints that must be met.

EASY PART: Calculus of maximizing / minimizing a cont (diff'ble) function $f: D \longrightarrow \mathbb{R}$ where $D = \underbrace{[a,b]}_{\text{bounded}}, \underbrace{[a,\infty), (-\infty,b] \text{ or } \mathbb{R}}_{\text{unbounded}}$

The domain D is determined by the constraints ($\mathbb{R}$ = no constraints)

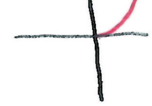
Roadmap: (1) Find the critical points ($f'(c)=0$ or $f'(c)$ does not exist)

(2) ———— values ($f(c)$ for c = crit pt.)

(3) Compute f at the endpoints of D (unless they are $\pm\infty$) & compare with (2).

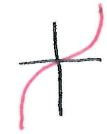
- If D is $[a,b]$ & f is cont, we know f has global max & min values by EVT
- In all 3 other cases, we need to study $\lim\limits_{x\to\infty} f(x)$ and/or $\lim\limits_{x\to-\infty} f(x)$ since f may not have max/min values (it won't if these limits are $\pm\infty$).

Examples $f(x) = x^2$ $f: [0, \infty) \longrightarrow \mathbb{R}$ has min, no max



$f(x) = x^3$, $f: \mathbb{R} \longrightarrow \mathbb{R}$ no max & no min

$f: [-\infty, b] \longrightarrow \mathbb{R}$ has max $f(b)$, no min $(f' \geq 0)$



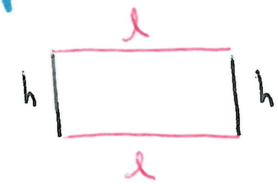
$f: [a, \infty] \longrightarrow \mathbb{R}$ has min $(f(a))$, no max

<u>Strategy for modeling</u>:

1. Understand the "word problem"
2. Make a careful sketch if problem is geometric. Don't include any unwanted assumptions (eg symmetries) that aren't there.
3. Label the figure with the given data & ~~suggestive~~ name variables. Write down any constraints
4. Write down the equations involved & what needs to be maximized/minimized. ⇝ Transform the eqn. into a single variable one (eg time)
5. Solve the max/min problem & make sure the answer <u>makes sense in the context</u>.

<u>Example 1</u>: Show that the rectangle with maximum area for a fixed perimeter is a square.

<u>Soln</u>



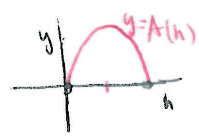
(1) $\underset{(l,h)}{\text{Area}} = l \cdot h$ ⇝ want to MAXIMIZE AREA.

(2) Perimeter $= 2l + 2h = 2(l+h) = P \geq 0$ is fixed

CONSTRAINTS: $\begin{cases} l, h \geq 0 \\ \cancel{2(l+h) = P} \end{cases}$

Use (2) to turn Area into a single variable expression.

How? Solve for l: $\boxed{l = \frac{P}{2} - h}$ & replace in Area.



$\text{Area} = \text{Area}(h) = \left(\frac{P}{2} - h\right) h = \frac{P}{2} h - h^2$ cont.

Constraints: $h \geq 0$, $\frac{P}{2} - h \geq 0$ ⇝ $\boxed{0 \leq h \leq \frac{P}{2}}$ so Domain Area $= [0, \frac{P}{2}]$

Conclusion: Need to maximize $A(h) \underset{\text{cont}}{=} \frac{P}{2}h - h^2$ for $0 \le h \le \frac{P}{2}$.

EVT ensures we have a max value, we just need to find it.

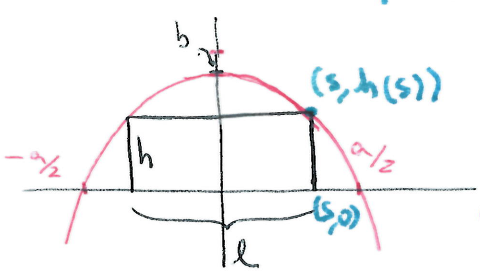
1. Crit pts: $A'(h) = \frac{P}{2} - 2h = 0$ so $h = \frac{P}{4}$
2. Crit Values: $A\left(\frac{P}{4}\right) = \frac{P}{2}\frac{P}{4} - \left(\frac{P}{4}\right)^2 = \frac{P^2}{8} - \frac{P^2}{16} = \frac{P^2}{16}$
3. Endpoints: $A(0) = 0 = A\left(\frac{P}{2}\right)$
4. Comparison gives $A\left(\frac{P}{4}\right)$ as the winner, so $\boxed{h = \frac{P}{4} \quad l = \frac{P}{2} - h = \frac{P}{4}}$

Example 2 Given 2 positive constraints a & b, consider the region between the parabola $a^2 y = a^2 b - 4bx^2$ and the x-axis. It is a parabolic segment of base a & height b. Find the base & height of the largest rectangle (in Area) with lower base on the x-axis & upper vertex on the parabola.

Soln: • Vertex of parabola $y = b - \frac{4b}{a^2}x^2$ is $y' = -\frac{8b}{a^2}X = 0$ so $x = 0$ & $y = b$.

• x-intercepts: $b - \frac{4b}{a^2}x^2 = 0 \rightsquigarrow ba^2 = 4bx^2$

$\frac{a^2}{4} = x^2 \rightsquigarrow x = \pm\frac{a}{2}$



We have a y-axis symmetry so $l = 2s$

(1) Area $= l\,h = 2s\,h(s)$

Here $h(s) = b - \frac{4b}{a^2}s^2$

So $\underset{(s)}{\text{Area}} = 2s\left(b - \frac{4b}{a^2}s^2\right) = 2bs - \frac{8b}{a^2}s^3$ (cont function of 1 variable s)

Constraints $= 0 \le s \le a/2$

Again: bounded problem where a max is known to exist by EVT.

1. Crit pts: $A'(s) = 2b - \frac{24b}{a^2}s^2 = 2b\left(1 - \frac{12}{a^2}s^2\right) = 0$ so $\underset{(b \ne 0)}{} s^2 = \frac{a^2}{12}$

$s = \frac{\pm a}{2\sqrt{3}}$ but $s \ge 0$ & $a \ge 0$ so we discard the neg soln $\rightarrow \boxed{s = \frac{a}{2\sqrt{3}}}$

2. Crit value: $A\left(\frac{a}{2\sqrt{3}}\right) = 2b\frac{a}{2\sqrt{3}} - \frac{8b}{a^2}\frac{a^3}{2^3\,3\sqrt{3}} = \frac{ba}{\sqrt{3}} - \frac{ba}{3\sqrt{3}} = \frac{2ba}{3\sqrt{3}}$

3. End points: $A(0)=0$ $\quad A\left(\frac{a}{2}\right)=0$ $\quad (h=0)$

4. Comparison: $\frac{a}{2\sqrt{3}}$ wins so $l=\text{base}=2.s=\boxed{\frac{a}{\sqrt{3}}}$

height $=b-\frac{4b}{a^2}\left(\frac{a}{2\sqrt{3}}\right)^2=b-\frac{b}{3}=\boxed{\frac{2b}{3}}$