Lecture XVI : § 4.4 More max-min problems.
§ 4.5 : Related Rates

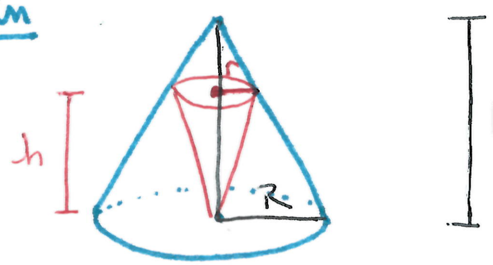
§1 More max-min problems

Last time: Discuss how to model (eqns + constraints) applied max / min problems

Keys ① Use the eqns to reduce to a function in one variable with constraints
② Find max / min via critical values & compare with end points' values
③ Check answers in context of the problem.

Example A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = H/3$.

Soln



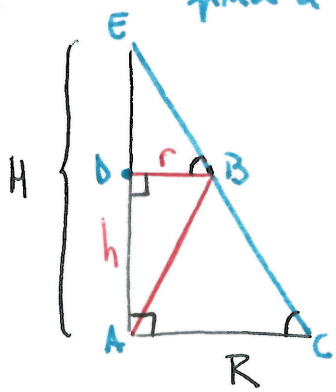
- Big cone : height $H \geq 0$, radius $R \geq 0$ } fixed
- Small cone : height $h \geq 0$, radius $r \geq 0$ } variables

inscribed = inside + touching the boundary

Constraints : $0 \leq h \leq H$ & $0 \leq r \leq R$

$\underset{(r,h)}{\text{Vol cone}} = \dfrac{\pi r^2 h}{3}$ → want to MAXIMIZE it.

Q How to find a relation between r & h? A: Use geometry of triangular cross section.



$DB \,//\, AC$ ("inscribed") so $D\hat{B}E = A\hat{C}E = \alpha$

So $\tan(\alpha) = \dfrac{H-h}{r} = \dfrac{H}{R}$

In particular : $\boxed{r = \dfrac{R}{H}(H-h)}$

Replace this in $\underset{(h)}{\text{Vol}} = \dfrac{\pi}{3}\,\dfrac{R^2}{H^2}(H-h)^2\, h$ cont & diff'ble.

Constraints : $0 \leq h \leq H$ [since $0 \leq \dfrac{R}{H}(H-h) \leq R \iff 0 \leq H-h \leq H$ so don't get new conditions]

Now, we solve the abstract problem.

- End pts : $\text{Vol}(0) = \text{Vol}(H) = 0$ so max is a crit pt.

• $\mathrm{Vol}'(h) = \frac{\pi}{3}\frac{R^2}{H^2}\left(2(H-h)(-1)\,h + (H-h)^2\right)$

$= \frac{\pi}{3}\frac{R^2}{H^2}(H-h)(-2h+H-h) = \frac{\pi}{3}\frac{R^2}{H^2}(H-h)(H-3h)$

So $\mathrm{Vol}'(h)=0$ for $h=H$ or $h=\frac{H}{3}$

• Crit values: $\mathrm{Vol}(H)=0$, $\mathrm{Vol}\left(\frac{H}{3}\right) = \frac{\pi}{3}\frac{R^2}{H^2}\left(\frac{2H}{3}\right)^2\frac{H}{3} = \frac{4\pi R^2 H}{81} > 0$

$\Rightarrow$ MAX for $\boxed{h=\frac{H}{3}}$.

§2 Related rates

• We have 2 or 3 quantities linked together by a constraint (typically, an equation)

• The system is changing ("in motion") and so all quantities are changing ("with time")

GOAL Compute the rate of change of one quantity in terms of the known rate of change of the rest.

• Tools: • implicit differentiation
  • chain rule + substitution of known values

• Only difficulty: modelling the problem

• Same strategy as with max/min problems

1. Draw diagram, label figures. Identify data & units of measurement
2. DON'T fix quantities that are changing until the very end.

• Only one Novelty: • Find the relationship between the varying quantities

Example 1: Air is being pumped into a spherical balloon whose volume increases at a fix rate of $8\ \mathrm{ft}^3/\mathrm{min}$. Find the rate of growth of the radius of the balloon when its volume is $1\ \mathrm{ft}^3$.

Soln: $V(t) = V(r(t)) = \frac{4}{3}\pi\, r(t)^3$

Know: $8 = \frac{dV}{dt} = \frac{4}{3}\pi\, 3\, r(t)^2\,\frac{dr}{dt}$ Want to solve for $\frac{dr}{dt}$

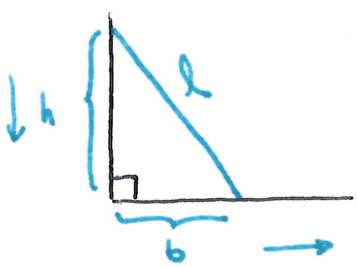
$\frac{dr}{dt} = \frac{8}{4\pi}\frac{1}{r^2} = \frac{2}{\pi r^2}$

Missing information $V = \frac{4}{3}\pi r^3 \rightsquigarrow r = \left(\frac{3}{4\pi} V\right)^{1/3}$

Replace this in $\frac{dr}{dt}$: $\frac{dr}{dt} = \frac{2}{\pi\left(\frac{3}{4\pi}V(t)\right)^{2/3}} = \frac{2}{\pi\left(\frac{3}{4\pi}1\right)^{2/3}} = \frac{4\sqrt[3]{2}}{\sqrt[3]{\pi}\sqrt[3]{9}}$

<u>A</u>: $\frac{dr}{dt} = \frac{4\sqrt[3]{2}}{\sqrt[3]{\pi}\sqrt[3]{9}}$ ft/min.

<u>Example 2</u>: Find the speed at which a ladder slides down a wall if it slides away from the wall at a fixed rate.



l = length of the ladder is fixed

$b = b(t)$ = distance to the wall

$h = h(t)$ = ——— floor

Equation: $\boxed{l^2 = b^2 + h^2}$ (*)

<u>Know</u>: $b'(t)$ <u>Want to compute</u> $h'(t)$.

Implicit diff of (*) gives $0 = 2bb' + 2hh'$ & we solve for h'

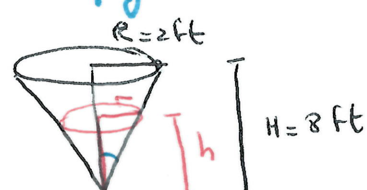
$$\boxed{h' = \frac{-bb'}{h}}$$

More concretely: if $l = 13$ ft, $b' = 6$ ft/min, $b = 5$ ft, then

$$h'(t) = \frac{-5\cdot 6 \text{ ft}^2/\text{min}}{\sqrt{13^2-5^2}\text{ ft}} = \frac{-30}{12}\text{ ft/min} = \frac{-5}{2}\text{ ft/min}.$$

<u>Example 3</u>: A conical water tank with its vertex down is 8 ft high & 4 ft in diameter at the top. The tank is full & water leaks through a whole in the bottom at a rate of 1 ft³/min. Find the rate at which the water level is falling when the tank is 7/8 empty.

$$\text{Vol} = \frac{\pi r^2 h}{3}$$



<u>Constraints</u>:

- Tank $\frac{7}{8}$ empty = $\frac{1}{8}$ full

<u>means</u> $\boxed{\frac{1}{8}H = h}$

$$\text{Full Vol} = \frac{\pi R^2 H}{3} = \frac{\pi\, 2^2 \cdot 8}{3} = \frac{32}{3}\pi \text{ ft}^3$$

- So we need t where $h(t) = \frac{8}{8} = \boxed{1}$

- Unknown : $h'(t)$
- Relation : $\frac{r}{h} = \frac{R}{H} = \frac{2}{8} \rightsquigarrow \boxed{r = \frac{1}{4} h}$

Replace in Vol: $\mathrm{Vol}(h(t)) = \dfrac{\pi \left(\frac{1}{4}h\right)^2 h}{3} = \dfrac{\pi\, h^3(t)}{3 \cdot 4^2}$

Impl. diff gives $\underbrace{\mathrm{Vol}'}_{-1\,ft^3/min} = \frac{\pi}{48} 3 h^2(t)\, h'(t) = \frac{\pi h^2}{16} h'$

So $-1 = \frac{\pi}{16} \cdot 1\, h'$ gives $\boxed{h' = \frac{-16}{\pi}\ ft/min}$

Example 4: Same data but want the rate at which the water level is falling when we've collected $36 \cdot \pi\, ft^3$ of water leaked.

- Now we have $1 \frac{ft^3}{min} = \frac{\pi h^2}{16} h'$.
- We know $\mathrm{Vol} = \frac{\pi\, h^3(t)}{48} = \pi\, 36 \rightsquigarrow h^3(t) = 36 \cdot 48 = 27 \cdot 4^3 = 12^3$

So $\boxed{h(t) = 12}$

We get $h' = -16/12^2\pi = \boxed{\frac{-1}{9\pi}\ ft/min}$