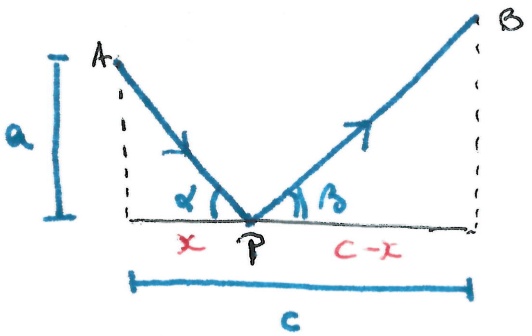
Lecture XVII: §4.4 Reflection & refraction

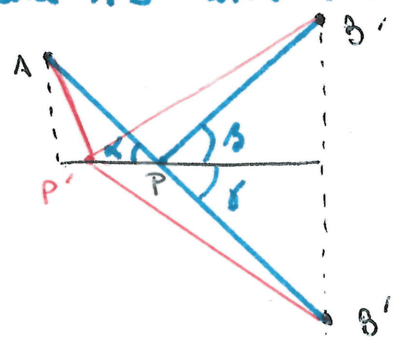
§1 Law of Reflection:



A ray of light travels from A to P, bounces ~~on~~ the mirror surfaces to reach B in the shortest amount of time

Prop: $\alpha = \beta$

Proof 1: Set B' = mirror image of B & draw P as the intersection of the line AB' with the mirror. Then $\alpha = \gamma$ & $\gamma = \beta$ shows $\alpha = \beta$.



Why is this the shortest path? (speed of light is constant so fastest way = shortest way).

TOTAL length $APB = AP + PB = AP + PB' = AB'$.

Now, for any other point P' in the mirror, draw the same picture: Length$(AP'B) = AP' + P'B = AP' + P'B'$

But $P'B' + AP' > AB'$ because straight line is shortest path between 2 points. Conclusion: shortest path between A & B passes through P & so $\alpha = \beta$ as we want.

Proof 2 Length of the path depends on the location of P. We set x to be the distance from P to the left. Then:

$$L(x) = AP(x) + P(x)B = \sqrt{x^2+a^2} + \sqrt{(c-x)^2+b^2}.$$

Constraints: $0 \leq x < c$

Note $a, b > 0$ so $L(x)$ is cont & differentiable by EVT it has a min

End points: $L(0) = a^2 + \sqrt{b^2+c^2}$ & $L(c) = \sqrt{c^2+a^2} + \sqrt{b^2}$.

Crit pts: $$L'(x) = \frac{2x}{2\sqrt{x^2+a^2}} + \frac{-2(c-x)}{2\sqrt{(c-x)^2+b^2}} = \frac{x}{\sqrt{a^2+x^2}} - \frac{(c-x)}{\sqrt{b^2+(c-x)^2}}$$

Note: $L'(0) = \frac{-c}{b} < 0$ & $L'(c) = \frac{c}{\sqrt{a^2+c^2}} > 0$ so L is decr near 0 & incr near c.

So $L'(x)=0$ gives $\dfrac{x}{\sqrt{a^2+x^2}} = \dfrac{c-x}{\sqrt{b^2+(c-x)^2}}$. & min must be a crit pt!

We square both sides: $\dfrac{x^2}{a^2+x^2} = \dfrac{(c-x)^2}{b^2+(c-x)^2}$

We invert: $\dfrac{a^2+x^2}{x^2} = \dfrac{b^2+(c-x)^2}{(c-x)^2} \rightsquigarrow \dfrac{a^2}{x^2}+1 = \dfrac{b^2}{(c-x)^2}+1$

so $\dfrac{(c-x)^2}{b^2} = \dfrac{x^2}{a^2} \rightsquigarrow$ we get a quadratic formula we can solve for x

We don't need to! Recall we want to show $\alpha=\beta$.

But $\tan\alpha = \dfrac{a}{x}$ & $\tan\beta = \dfrac{b}{c-x}$ so since $\left(\dfrac{a}{x}\right)^2 = \left(\dfrac{b}{c-x}\right)^2$ & $a,b,x,c-x \geq 0$

we get $\alpha=\beta$ at the crit pt! & $\alpha,\beta \in [0,\frac{\pi}{2}]$

. The crit pt gives the min $L(x)$. so $\alpha=\beta$ at that point.

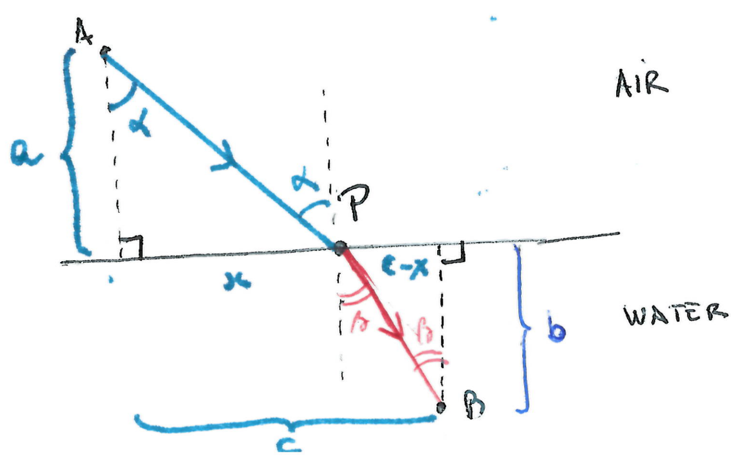
· Alternatively: use 2^{nd} derivative Test:

$$L'' = \frac{a^2}{(a^2+x^2)^{3/2}} + \frac{b^2}{(b^2+(c-x)^2)^{3/2}} > 0 \quad \text{(sum of positive numbers).}$$

§2 Law of Refraction:

Q: What happens when the ray of light travels through 2 different media (air & water/glass, ...)?

Again, we want to minimize the time it takes to travel from A (in one medium) to B (in a different medium).



Write v_a = speed of light in air (>0)

v_b = ———————— water (>0)

x = distance from P to left

so $\boxed{0 \leq x \leq c}$

Write 2 relevant angles $= \alpha$ & β (they come in pairs)

Snell's Law: $\dfrac{\sin(\alpha)}{\sin(\beta)}$ = constant. (called index of refraction & it only depends on v_a & v_b, not on the position of A & B)

Proof: We set this as a problem of minimizing the time it takes to travel the distance from A to B.

- Time in the air: $T_{air}(x) = \dfrac{\text{dist}}{|\text{velocity}|} = \dfrac{AP}{v_a} = \dfrac{\sqrt{a^2+x^2}}{v_a}$
- Time in the water: $T_{water}(x) = \dfrac{\text{dist}}{|\text{velocity}|} = \dfrac{PB}{v_b} = \dfrac{\sqrt{b^2+(c-x)^2}}{v_b}$

So Total Time $T(x) = T_{air}(x) + T_{water}(x)$

$$\boxed{T(x) = \frac{(a^2+x^2)^{1/2}}{v_a} + \frac{(b^2+(c-x)^2)^{1/2}}{v_b}}$$

Want to minimize $T(x)$ subject to $0 \leq x \leq c$

As before: T is continuous (& diff'ble) so by EVT we have a minimum.

- End points $T(0) = \dfrac{a}{v_a} + \dfrac{\sqrt{b^2+c^2}}{v_b}$; $T(c) = \dfrac{\sqrt{a^2+c^2}}{v_a} + \dfrac{b}{v_b}$.
- Crit pts: $T'(x) = \dfrac{1}{v_a}\dfrac{x}{\sqrt{a^2+x^2}} + \dfrac{1}{v_b}\dfrac{(-(c-x))}{\sqrt{b^2+(c-x)^2}}$

$T'(0) = \dfrac{-c}{v_b\sqrt{b^2+c^2}} < 0$ $T'(c) = \dfrac{c}{v_a\sqrt{a^2+c^2}} > 0$ so it's decr. near 0 & incr. near c. This forces the minimum to be a crit pt

Note: $T'(x) = 0$ gives $\dfrac{1}{v_a}\underbrace{\boxed{\dfrac{x}{\sqrt{a^2+x^2}}}}_{=\sin\alpha(x)} = \dfrac{1}{v_b}\underbrace{\boxed{\dfrac{c-x}{\sqrt{b^2+(c-x)^2}}}}_{\sin\beta(x)}$

So at the min, we have $\dfrac{\sin\alpha}{\sin\beta} = \dfrac{v_a}{v_b}$ = constant.

Alternatively: We can use 2nd Derivative Test

$T''(x) = \dfrac{1}{v_a}\dfrac{a^2}{(a^2+x^2)^{3/2}} + \dfrac{1}{v_b}\dfrac{b^2}{(b^2+(c-x)^2)^{3/2}} > 0$ so crit value is a min.