Lecture XIX: §5.3 Indefinite integrals & Substitution
§5.4 Differential Equations. Separation of variables

§1 Antiderivatives:

Up to now, we've discussed rules to find f' given a function f.

IDEA: Want to reverse the process ("antiderivation"), meaning "given $f(x)$, can we find $F(x)$ with $F' = \frac{dF}{dx} = f$?

Examples:
- $\frac{dF}{dx} = 3x^2 \rightsquigarrow dF = 3x^2\,dx \rightsquigarrow F = \underline{x^3},\ \underline{x^3}+1, \ldots$
- $\frac{dF}{dx} = \cos(x) \rightsquigarrow dF = \cos(x)\,dx \rightsquigarrow F = \sin(x),\ \sin(x) - \frac{\pi}{4}\sqrt{2}, \ldots$

Guiding question: Can we recover a function from its derivative? Almost!

Proposition: If $F(x)$ & $G(x)$ are two functions with the same derivative $f(x)$ on an interval $[a,b]$ with $a<b$, then $F(x)-G(x)$ is constant in (a,b).

Why? $\frac{d}{dx}(F(x)-G(x)) = f(x) - f(x) = 0$ so $F(x)-G(x)$ is a diff'ble function with derivative $=0$. By Mean Value Thm we conclude that $F(x)-G(x)$ is constant in any $[c,d]$ with $a<c<d<b$, so $F(x)-G(x)$ is constant in (a,b).

$\left(0 = f'(\xi) = \frac{f(d)-f(c)}{d-c} \rightsquigarrow f(d) = f(c)\right)$ □

Conclusion: Antiderivatives are not unique, but they all differ by constants. So if $F(x)$ has $F'(x) = f$, all other antiderivatives have the form $F(x)+C$ for C an arbitrary constant.

Notation: $\int f(x)\,dx = F(x) + C$ Call it indefinite integral/antiderivative.

Remark: Every formula for derivatives has an analog for indefinite integrals ("Derivative rules for differentials" discussed last time)

① Power Rule: $\frac{d}{dx}x^n = nx^{n-1} \rightsquigarrow dx^n = nx^{n-1}\,dx \rightsquigarrow x^{n-1}\,dx = \frac{dx^n}{n} = d\left(\frac{x^n}{n}\right)$

Then $\int nx^{n-1}\,dx = x^n + C$

So $\boxed{\int x^m\,dx = \frac{x^{m+1}}{m+1} + C}$ for all m rational $\boxed{m \neq -1}$

② Trig Rules $\int \sin x\, dx = -\cos x + C$; $\int \cos(x)\, dx = \sin(x) + C$

$\int \sec^2(x)\, dx = \tan(x) + C$; $\int \csc^2(x) = -\cot(x) + C$

$\int \underbrace{\tan^2 \sec(x)}_{= \frac{\sin(x)}{\cos^2(x)}}\, dx = \sec(x) + C$; $\int \cot^2(x) \csc(x) = -\csc(x) + C$

③ Multiplication by constants & additivity:

- a constant $\frac{d}{dx}(aF(x)) = a\frac{dF}{dx}$ so $\int a f(x)\, dx = a\int f(x)\, dx$.
- $d(F+G) = F'(x)\, dx + G'(x)\, dx$ so $\underbrace{\int f(x) + g(x)\, dx}_{F+G} = \underbrace{\int f(x)\, dx}_{=F} + \underbrace{\int g(x)\, dx}_{G}$

⇝ Get antiderivatives of polynomials!

Eg: $\int 3x^2 + 4x + 2\, dx = 3\int x^2\, dx + 4\int x\, dx + 2\int 1\, dx$

$= 3\left(\frac{x^3}{3} + C_1\right) + 4\left(\frac{x^2}{2} + C_2\right) + 2(x + C_3)$

$= x^3 + 2x^2 + 2x + \underbrace{(3C_1 + 4C_2 + 2C_3)}_{= C \text{ arbitrary const!}}$

Also, any powers except $\frac{1}{x} = x^{-1}$.

Eg $\int x^{1/3}(x+2)^2\, dx = \int x^{1/3}(x^2 + 4x + 4)\, dx = \int x^{7/3} + 4x^{4/3} + 4x^{1/3}\, dx$

$= \frac{x^{10/3}}{10/3} + \frac{4x^{7/3}}{7/3} + \frac{4x^{4/3}}{4/3} + C = \frac{3}{10}x^{10/3} + \frac{12}{7}x^{7/3} + 3x^{4/3} + C$

④ Most subtle techniques = Chain Rule ⇝ $\boxed{\text{Substitution}}$

(Other rules ⇝ future lectures (integration by parts, ...))

§2 Substitution:

Idea: $dF = f(x)\, du$

If $f = u^{1/2}$, then $\int u^{1/2}\, du = \frac{2}{3}u^{3/2} + C$

What if $u = u(x) = x^2 + 1$ then $du = u'\, dx = 2x\, dx$ &

$dF = f(u)\, du = \sqrt{x^2+1}\, 2x\, dx = f(u)\, u'(x)\, dx$,

(Chain Rule)

Reverse the process:

$\int \sqrt{x^2+1}\, 2x\, dx = \int \sqrt{u}\, du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^2+1)^{3/2} + C$

Substitute: $u = x^2 + 1$, $du = 2x\, dx$; Subst back

Check: $\frac{d}{dx}\left(\frac{2}{3}(x^2+1)^{3/2}\right) = \sqrt{x^2+1}\, 2x\, dx$ by Chain Rule

Substitution Rule If $F_{(x)} = h(u(x))$, write $F = h(u)$ & $u = u(x)$

Then $dF = h'(u)\,du$ & $du = u'(x)\,dx$ so $dF = h'(u(x))\,u'(x)\,dx$ &

$$\left(C + F_{(x)} = \int dF =\right) \int h'(\underbrace{u(x)}_{=u})\, \underbrace{g'(x)\,dx}_{=du} \underset{\text{substitution 1}}{=} \int h'(u)\,du = h(u) + C$$

$$\underset{\text{subst 2}}{=} h(u(x)) + C = F_{(x)} + C$$

Q: How do we use this?

Say we want to solve $\int \sqrt{x^2+1}\; 2x\,dx$.

1°) We recognize $(x^2+1)' = 2x$ & set $u = x^2+1$, $du = 2x\,dx$

2°) Substitution gives $\int \sqrt{x^2+1}\;\underbrace{2x\,dx}_{du} = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = F(u)$

3°) __Always__ put back the result in terms of the original variables, x in this example, so $\int \sqrt{x^2+1}\;2x\,dx = \frac{2}{3}(x^2+1)^{3/2} + C$.

NOTES ① · Integration involves 2 substitutions
· Can check the result with Chain Rule

② Rule is fairly straight-forward if we view the differentiation step as computing a differential.

Practice Helps!

→ ③ Finding the expression $u = u(x)$ can be tricky! We must be able to write the integrand $f(x)\,dx$ as $h(u)\,du$ making sure NO x remains!

Examples: ① $\int \cos^5(x)\sin x\,dx \underset{\substack{u = \cos(x)\\ du = -\sin(x)\,dx}}{=} \int u^5(-du) = -\int u^5\,du = \frac{-u^6}{6} + C$

$$= \boxed{\frac{-\cos^6 x}{6} + C.}$$

② $\int \sin(5x)\,dx \underset{\substack{u = 5x\\ du = 5\,dx}}{=} \int \sin(u)\,\frac{du}{5} = \frac{1}{5}\int \sin u\,du = \frac{-\cos u}{5} + C$

$$= \boxed{\frac{-\cos 5x}{5} + C}$$

③ $\int \frac{dx}{(x-7)^7} \underset{\substack{u = x-7\\ du = dx}}{=} \int \frac{1}{u^7}\,du = \int u^{-7}\,du = \frac{u^{-6}}{-6} + C = -\frac{(x-7)^{-6}}{6} + C$

$$= \boxed{\frac{-1}{6(x-7)^6} + C}$$

§3 Differential equations & separation of variables

The theory of diff'l equations aims to recover a function from a relation among its derivatives.

Examples: ① $y' = 2x$ ⇝ By integration: $dy = 2x\,dx$ so
$$y = \int dy = \int 2x\,dx = \boxed{x^2 + C}$$ for any constant C.

② $y'' = -y$ or $y'' + y = 0$ ⇝ By inspection $y(x) = a\sin x + b\cos x$ for constants a, b.

We can check it: $y' = a\cos x - b\sin x$
$y'' = -a\sin x - b\cos x = -y$ so they are solns!

Harder: show there are no other solutions.

③ $y''' + x^2 y'' - y = 0$ ⇝ Soln = ???

Certain simple O.D.E. (ordinary diff'l eqns) can be solve via the method of separation of variables (analog of implicit differentiation)

Example $y' = -x^2y^2$ ⇝ write $\frac{dy}{dx} = -x^2y^2$ or $dy = -x^2y^2dx$

STEP 1: We put different variables on different sides of the equation ("separation step")
$$-\frac{dy}{y^2} = +x^2\,dx$$

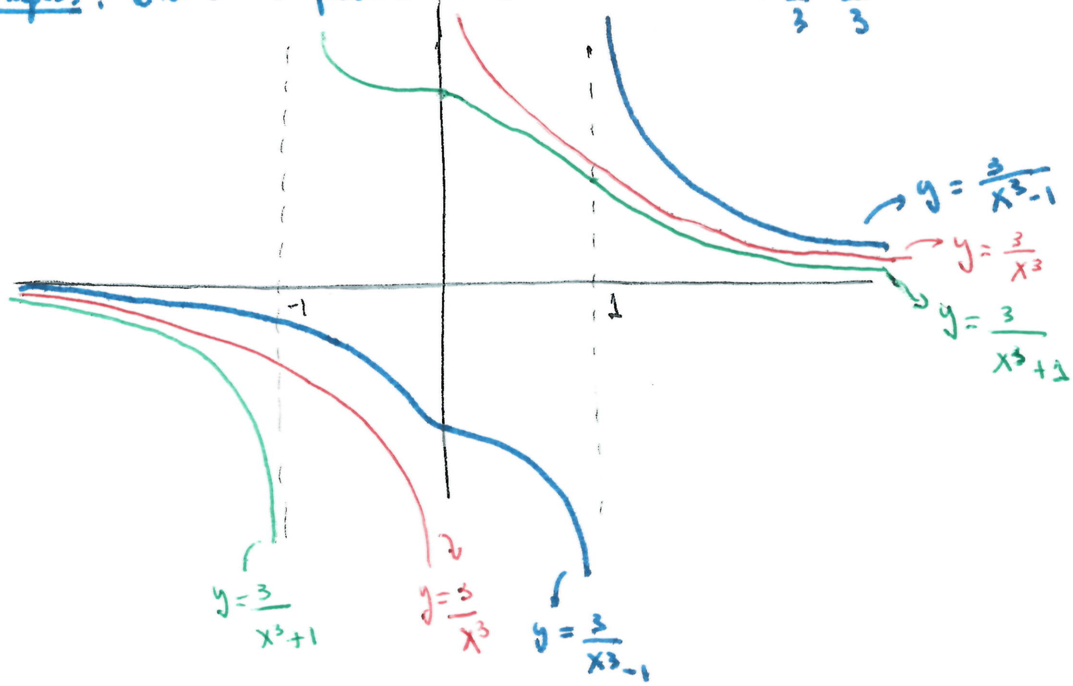
STEP 2: Integrate both sides
$$\frac{1}{y} = \int -y^{-2}dy = \int \frac{-dy}{y^2} = \int +x^2\,dx = \frac{x^3}{3} + C$$
C = arbitrary constants

STEP 3 Solve for y. ⇝ $\boxed{y = \frac{1}{\frac{x^3}{3} + C}}$ This is the form of a general solution

Notice that we get a one-parameter family (C) of solutions

A particular solution = choose a value of C.

Examples: Draw 3 particular solutions $C = 0, \frac{1}{3}, \frac{-1}{3}$.



Q: How do we determine C?

A We need to impose an initial condition, that is a value $y(x_0)$ for some x_0

Ex $y(0) = 3 = \frac{1}{0+C}$ gives $C = \frac{1}{3}$.

Note: Separation of variables only works for very special ODEs, namely

- equation only involves y', y, x (1^{st} order)
- need to be able to split the equation into $g(y)\,dy = f(x)\,dx$.

Non-example: $y' = \frac{x+y}{x-y}$.

Example: $\begin{cases} y' = 2y^2(4x^3 + 4x^{-3}) \\ y(0) = 1 \quad \text{(initial condition)} \end{cases}$ $\rightsquigarrow \frac{dy}{y^2} = (8x^3 + 8x^{-3})\,dx$

So $-y^{-1} = 2x^4 - 4x^{-2} + C$ $\qquad y = \frac{1}{-2x^4 + 4x^{-2} - C}$ general soln

Soln has $y(0) = 1$ so $1 = \frac{1}{0 - C}$ gives $C = -1$

Soln becomes $\boxed{y(x) = \frac{1}{-2x^4 + 4x^{-2} + C}}$ (particular soln given the initial conditions)