Lecture XX: §5.5 Motion under gravity

Recall: Position of a particle (with respect to time) $= s(t)$ $\quad (t \geq 0)$

Velocity $= v(t) = s'(t)$

Newton's Law of Motion:

I. A particle in a state of rest of motion will continue to be so unless an external force is applied to it.

II. $a(t) =$ acceleration $= s''(t)$ $\quad$ Then $a(t) = \dfrac{F}{m}$, equivalently

$m =$ mass

$F =$ force

$$\boxed{F = ma = m s''(t)} \quad (*)$$

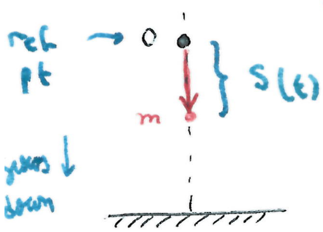
Ordinary diff'l equation.

III. To every action, there's an equal and opposite reaction.

GOAL: Solve $(*)$. Since we have s'' involved we'll need 2 initial conditions (eg $s(0)$, $s'(0)$) to give a particular solution.

Examples: Gravity force induces a constant acceleration $\boxed{g \approx 9.8 \frac{m}{s^2} = 32 \, ft/s^2}$

Problem Find the motion of a stone of mass m which is dropped from a point above the surface of the earth.



Initial conditions: $s(0) = 0$ (don't know initial height so this is our reference pt)

$v(0) = 0$ (stone is dropped)

Eqn: $m\, a(t) = \boxed{F = mg}$ gravitational force, no resistance from air.

$\rightsquigarrow$ $a(t) = s''(t) = g$ constant.

Solve for s by integrating twice

- $v'(t) = a(t)$ so $v(t) = \int g \, dt = gt + C_1$.

  Initial cond $v(0) = 0$ yields $C_1 = 0$ so $v(t) = gt$

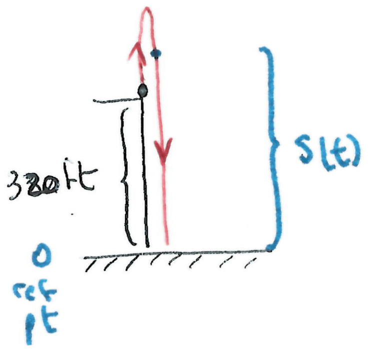
- $s'(t) = v(t)$ so $s(t) = \int gt \, dt = g\dfrac{t^2}{2} + C_2$

  Initial condition $s(0) = 0$ yields $C_2 = 0$ $\quad \boxed{\text{Soln} = s(t) = \frac{1}{2} g t^2}$

In general: $s(t) = \frac{1}{2} g t^2 + c_1 t + c_2$ for c_1, c_2 constants

These are determined by 2 "independent" initial conditions

Problem 2: Assume a stone is thrown upwards at 128 ft/s from the roof of a building 320 ft high. Find ~~the~~ Trajectory and determine its maximal height and at what time does the stone hit the ground



Initial conditions: $s(0) = 320$

$s'(t) = 128$ (goes up so sign is +)

$F = -mg = s''(t)\, m \rightsquigarrow \boxed{s''(t) = -g}$

gravity falls down and height grows in opposite direction

Again, we solve by integration:

- $v(t) = \int s''(t)\, dt = \int -g\, dt = -gt + C_1$
- $s(t) = \int v(t)\, dt = \int (-gt + C_1)\, dt = -g\frac{t^2}{2} + C_1 t + C_2$

Use initial conditions: $\left.\begin{array}{l} s(0) = C_2 = 320 \\ s'(0) = C_1 = 128 \end{array}\right\}$ $\boxed{s(t) = -16t^2 + 128t + 320}$

- Maximal height = $s'(t) = 0$ so $s'(t) = v(t) = -32t + 128 = 0$

  gives $t = \frac{128}{32} = 4 > 0$

  - height = $s(4) = -16\, 4^2 + 128 \cdot 4 + 320 = \boxed{576 \text{ ft}}$

- When is $s(t) = 0$? A Solve for t using the quadratic eqn: $t = 10$ & $t = -2$

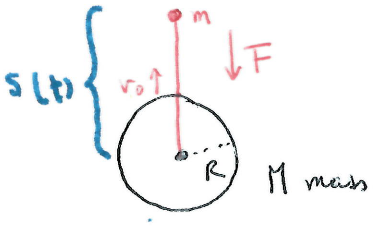
  Since $t \geq 0$ we must hit the ground after $\boxed{10 \text{ seconds}}$

- Speed? $|v(10)| = |-32 \cdot 10 + 128| = |-192| = \boxed{192 \text{ ft/s}}$

Method:
- Integrate twice from $a(t) = \pm g$
- Use initial conditions to get values of 2 constants c_1, c_2.

§2 Escape Velocity

GOAL: Determine the initial velocity v_0 we should vertically fire a rocket for it to come to rest & escape completely from the earth's gravitational attraction.



Assumption: earth = particle of mass M located at its center

When firing the rocket: gravitational force will depend on the distance from the rocket to the center of the earth.

Newton's Law of gravitation: Any 2 particles of matter attract each other with a force that is jointly proportional to their masses & inverse proportional to the square of the distance between them.

$$\boxed{F = -G\,\frac{M\cdot m}{s^2}}$$

direction of movement

$G>0$ constant
s = distance, M, m masses

We get
$$m\,a(t) = \frac{-G\,M\,m}{s(t)^2}$$
(rocket's perspective) (gravitational force)

so
$$\boxed{s''(t) = \frac{-GM}{s(t)^2}} \quad (*)$$

Note: On the ground: $s''(0) = -g$ & $s(0) = R$

From this we get $-g = \dfrac{-GM}{R^2}$ so $\boxed{-GM = -gR^2}$ & we can replace this in $(*)$ to get

$$\boxed{v'(t) = s''(t) = \frac{-gR^2}{s^2(t)}}$$

We want to solve for $v(t)$! Escape velocity = initial velocity $v(0) = v_0$ (TBD)

Trick: Think of s as a variable & use chain rule $\dfrac{dv}{dt} = \dfrac{dv}{ds}\,\dfrac{ds}{dt} = \dfrac{dv}{ds}\cdot v$

So we get $\dfrac{dv}{ds}\,v = \dfrac{-gR^2}{s^2} \rightsquigarrow \underbrace{v\,dv}_{(v\text{-side})} = \underbrace{\dfrac{-gR^2}{s^2}\,ds}_{(s\text{-side})}$ Variables are separated!

Integrate both sides: $\frac{v^2}{2} = \int v\,dv = \int \frac{-gR^2}{s^2}\,ds = g\frac{R^2}{s} + C$

Q1: What's C? Use initial conditions $v(0) = v_0 \rightsquigarrow$ this is what we want to find!
$s(0) = R$

$$\frac{v_0^2}{2} = g\frac{R^2}{R} + C \qquad \text{so} \quad C = \frac{v_0^2}{2} - gR$$

Soln: $\boxed{\frac{v^2(t)}{2} = g\frac{R^2}{s(t)} + \left(\frac{v_0^2}{2} - gR\right)}$

Q2: How to escape gravitational force? A: We need $v(t) > 0$ for all t.

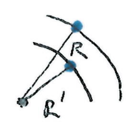
- Note: $g\frac{R^2}{s(t)} \xrightarrow[t\to+\infty]{} 0^+$ if we escape earth ($s(t)$ will go to $+\infty$.)
- Only way to ensure $v(t) > 0$ for all t is to require $\frac{v_0^2}{2} - gR \geq 0$

Since $v_0 > 0$, this gives $\boxed{v_0 \geq \sqrt{2gR} = \text{Escape Velocity}} = \sqrt{\frac{2GM}{R}}$

$\left(g = \frac{GM}{R^2}\right)$

- Values for earth: $R = 4000$ mi $\rightsquigarrow \sqrt{2gR} \simeq 7$ mi/s
$g = 32\ \text{ft}/s^2 = \frac{32}{5280}\ \text{mi}/s^2$
- Same formula works for any planet (different values of G, M, R)

Application: If the mass M is preserved, but R decreases to R', then escape velocity increases! $\left(= \sqrt{\frac{2GM}{R'}} > \sqrt{\frac{2GM}{R}} \right)$



So if escape velocity > speed of light, the light can never escape the gravitational force. This explains black holes!