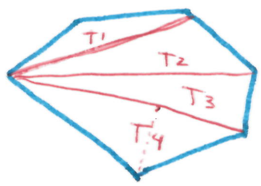
Lecture XXI: §6.2 The Problem of Areas

GOAL: Find a method to compute areas of regions enclosed by simple curves

§1 The Problem of Areas:

Example 1

Closed convex Polygons



1. Triangulate it from one vertex
2. Add up the areas of the Δs

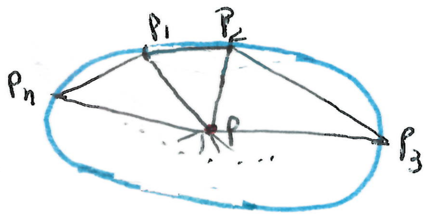
\# Triangles = \# sides -2

Q: Other non linear convex regions?

A: Method of exhaustion.

1. Pick a finite number of points in the boundary: $P_1, \ldots, P_n$ (clockwise)
2. Fix a point P in the interior
3. Build all triangles $\widehat{PP_1P_2}$, $\widehat{PP_2P_3}$, $\ldots$, $\widehat{PP_{n-1}P_n}$, $\widehat{PP_nP_1}$

region $= S$



Convex: All triangles are inside S

· Triangles don't overlap by construction (we are triangulating the polygon with vertices $P_1, \ldots, P_n$ using the point P as it's center.

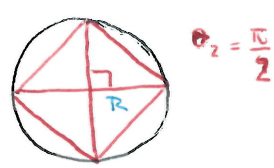
4. Make $n \to \infty$ & place the points $P_1, \ldots P_n$ close to each other.

The triangles cover almost all of the region S, so they approximate Area(S).

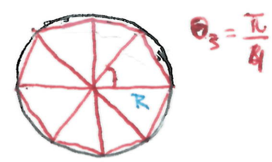
Example 2 S = circle of radius R, $P = (0,0)$

Choose $P_1, \ldots, P_{2^N}$ as the vertices of a regular 2^N-gon. Call it S_N

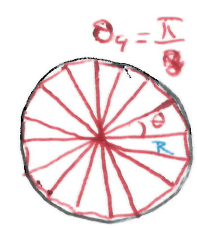
$n = 2^N$

$S =$ 

$N=2$

$S =$ 

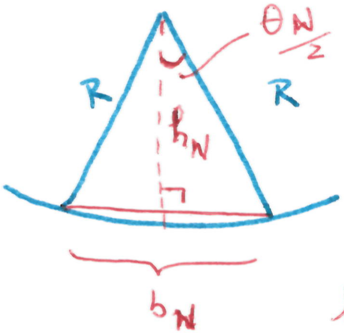
$N=3$

$S =$ 

$N=4$

Q: Why 2^n?

- At each step: we divide the angle θ by 2 so get $\theta_N = \frac{2\pi}{2^N} = \frac{\pi}{2^{N-1}}$
- ——— : S_N is contained in S_{N+1} (square $\subseteq$ octagon $\subseteq \ldots$)
- Easy area calculation: Area $(S_N) = 2^N$ Area of 1 triangle



$h_N = R\cos\frac{\pi}{2^N} \xrightarrow[N\to\infty]{} R$

$$\text{Area } \Delta = \frac{h_N \cdot b_N}{2}$$

$$\text{So Area}(S_n) = 2^N \frac{h_N b_N}{2} = \frac{h_N}{2}\big(\underbrace{b_N + \dots + b_N}_{2^N \text{ times}}\big)$$

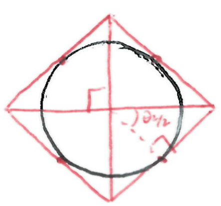
$$= \frac{h_N}{2} \text{ Perimeter}(S_N)$$

Q: What happens as we exhaust the circle? ($N \to \infty$)

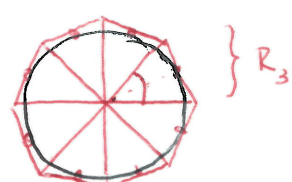
- Perimeter $(S_n) \xrightarrow[n\to\infty]{}$ Perimeter circle $=$ circumference $= 2\pi R$
- $h_n \xrightarrow[n\to\infty]{} R =$ radius of the circle

So Area(circle) $\overset{(*)}{=} \lim_{n\to\infty} \text{Area}(S_N) = \frac{R}{2}\, 2\pi R = \boxed{\pi R^2}$ □

Q: What if we had taken circumscribed 2^N-gons rather than inscribed ones?

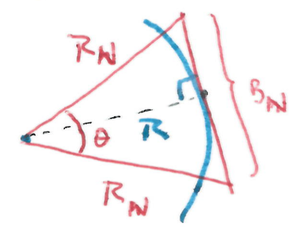


$N=2$



$N=3$

Close up on each Δ =



tangencies occur at the midpoint of each side of the 2^N-gon

Outer side has length B_N

$2 \cdot R\tan\frac{\theta}{2} = 2R\tan\frac{\pi}{2^N}$

$$\text{Area } \Delta = \frac{R \cdot 2R\tan(\pi/2^N)}{2} = R^2\tan\left(\frac{\theta_N}{2}\right)$$

$$\text{Area (Outer } S_n) = 2^N \cdot R^2 \tan\frac{\theta_N}{2}$$

$$= \frac{R}{2}\big(\underbrace{2R\tan\frac{\theta_N}{2} + \dots + 2R\tan\frac{\theta_N}{2}}_{2^N \text{ times}}\big)$$

$$= \frac{R}{2} \text{ Outer Perimeter.}$$

Outer Perim $\to 2\pi R$

We get $\boxed{\text{Area(Inner Polygon)} \leq \text{Area(Circle)} \leq \text{Area(Outer Polygon)}}$ $(*)$

$$\underbrace{\frac{h_n}{2}\ \text{Inner Perim}}_{\downarrow\ \pi R^2} \leq \text{Area(Circle)} \leq \underbrace{\frac{R}{2}\ \text{Outer Perim.}}_{\downarrow\ \pi R^2}$$

- So knowing the Perimeter gives the formula for the Area.
- We conclude by "Squeeze Lemma" that Area Circle $= \pi R^2$ (This justifies $=$ in $(*)$)
- We can use this to approximate π up to any digit we want!

How? Set $R=1$

Recall
- $h_N = \cos\left(\frac{\pi}{2^N}\right)$
- $b_N = 2\,\text{sen}\left(\frac{\pi}{2^N}\right) \rightsquigarrow$ Inner Perim $= 2^{N+1}\,\text{sen}\left(\frac{\pi}{2^N}\right)$
- $B_N = 2\tan\left(\frac{\pi}{2^N}\right) \rightsquigarrow$ Outer Perim $= 2^{N+1}\tan\frac{\pi}{2^N}$

So Outer Area $= 2^N \tan\frac{\pi}{2^N}$

Inner Area $= \frac{2^{N+1}}{2}\,\text{sen}\left(\frac{\pi}{2^N}\right)\cos\left(\frac{\pi}{2^N}\right) = 2^N\,\text{sen}\frac{\pi}{2^N}\cos\frac{\pi}{2^N} \underset{1}{=} 2^{N-1}\,\text{sen}\left(\frac{\pi}{2^{N-1}}\right)$

$\text{sen}(2x) = 2\cos x\,\text{sen}\,x$

$x = \frac{\pi}{2^N}$.

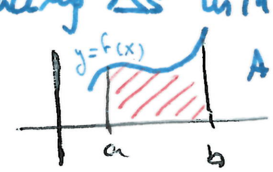
We replace these 2 expressions in $(*)$

$$\text{Inner Area} \leq \text{Area}(\bigcirc) \leq \text{Outer Area}$$

$$\begin{array}{c} \| \\ 2^{N-1}\,\text{sen}\left(\frac{\pi}{2^{N-1}}\right) \end{array} \leq \pi \leq 2^N \tan\frac{\pi}{2^N}$$

If the ends agree in the first k decimal places (for N large enough), then these will give the first k digits of π.

- Why inscribed vs circumscribed? We'll use inner & outer approximations to compute areas under the graph of a positive function, replacing Δs with rectangles. This will give rise to Riemann Sums



$A = \int_a^b f_{(x)}\,dx$