Lecture XXII §6.3 Summation notation & some sums
§6.4. The Area under a curve. Definite Riemann Integrals

Last Time: we discussed the problem of areas

Idea: approximate convex regions by suitable triangulations.

: 2 notations for sums: S → Σ Greek S; S → $\int$ integral sign → Next Time

§1 Summation Σ

Notation: $a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$

Example: $\sum_{k=1}^{5} k = 1+2+3+4+5 = 15 = \frac{5\cdot 6}{2}$

$\sum_{j=2}^{8} j^2 = 4+9+16+25+36+49 = 139 = \frac{7\cdot 8\cdot 15}{6} - 1$

$\sum_{i=1}^{4} (-1)^i \frac{1}{i} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{-7}{12}$

We would like to have a closed formula for computing these sums whenever possible. This will help us compute Riemann Sums.

Prop 1 $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof 1:

$$\begin{array}{r} 1 + 2 + \cdots + n \\ + \quad n + n-1 + \cdots + 1 \\ \hline \underbrace{(1+n) + (1+n) + \cdots + (1+n)}_{n \text{ Times}} \end{array}$$

Add 'column by column.'

Conclusion: $2\sum_{k=1}^{n} k = (1+n)n$ so $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof 2 (This will work for other sums)

$(k+1)^2 = k^2 + 2k + 1$ so $\boxed{(k+1)^2 - k^2 = 2k+1}$ (*)

We can do a telescoping sum: $(a_2 - a_1) + (a_3 - a_2) + \cdots + (a_N - a_{N-1}) = a_N - a_1$ last first

In our case: $a_k = k^2$ $\quad (2^2-1^2)+(3^2-2^2)+\cdots+((n+1)^2-n^2) = (n+1)^2-1^2 = \boxed{n^2+2n}$

On the other hand, using (*), we can write (LHS) as:

$$(2\cdot 1+1)+(2\cdot 2+1)+\cdots+(2(n)+1) = \sum_{k=1}^{n}(2k+1)$$

By regrouping we get $= \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1 = \boxed{2\sum_{k=1}^{n} k + 1\cdot n}$.

So $n^2+2n = 2\sum_{k=1}^{n} k + n$ gives $n^2+n = 2\sum_{k=1}^{n} k$

& $\sum_{k=1}^{n} k = \frac{n^2+n}{2} = \frac{(n+1)n}{2}$.

Prop 2 $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Proof: $(k+1)^3 - k^3 \underset{\text{Binomial Thm}}{=} k^3+3k^2+3k+1-k^3 = \boxed{1+3(k^2+k)}$

So $\sum_{k=1}^{n} (k+1)^3 - k^3 = (n+1)^3 - 1^3$ (its Telescopic!)

$$\parallel$$

$$\sum_{k=1}^{n} 3(k^2+k)+1 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n$$

But we know a formula for $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

So $3\sum_{k=1}^{n} k^2 = (n+1)^3 - 1 - n - 3\sum_{k=1}^{n} k = (n+1)^3 - 1 - n - \frac{3n(n+1)}{2}$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}\left(2((n+1)^3-1-n) - 3n(n+1)\right) = \frac{1}{6}\left(2n^3+6n^2+6n+2-2-3n^2-3n\right)$$

$$= \frac{1}{6}(2n^3+3n^2+n)$$

$$= \frac{n}{6}(2n^2+3n+1) = \frac{n}{6}(n+1)(2n+1)$$

Q: How to extend this construction? $\rightsquigarrow$ HW5

Say we want to find a formula for $\sum_{k=1}^{n} k^m$ $\quad (m=1,2,3,\dots)$

1. Use Binomial Thm.:

$$(k+1)^{m+1} - k^{m+1} = (m+1)k^{m} + \binom{m+1}{2}k^{m-1} + \cdots + \binom{m+1}{m}k + 1$$

Add up both sides:

$$\text{(LHS)}: \sum_{k=1}^{m} (k+1)^{m+1} - k^{m+1} = (n+1)^{m+1} - n^{m+1}$$

$$\text{(RHS)} = (m+1)\boxed{\sum_{k=1}^{m} k^{m}} + \binom{m+1}{2}\sum_{k=1}^{n} k^{m-1} + \cdots + \binom{m+1}{m}\sum_{k=1}^{n} k + 1\cdot n$$

WHAT WE WANT!

what we know!

2. Use formulas for smaller exponents to get formula for $\sum_{k=1}^{n} k^{m}$

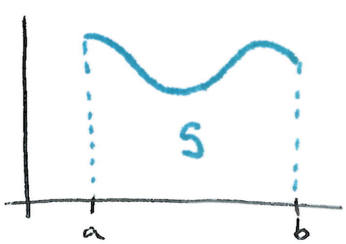
Eg $m=2$ used exp $=1$ formula

$m=3$ uses exp $=1$ & exp $=2$ formula. (HW5)

Prop 3: $\displaystyle\sum_{k=1}^{n} k^{3} = \frac{n^{2}(n+1)^{2}}{4}$

§2 The area under a curve

GOAL: Given a (cont.) function $f:[a,b] \to \mathbb{R}_{\geq 0}$, want to find the area of the region S that lies under the curve $y=f(x)$ & above the x-axis



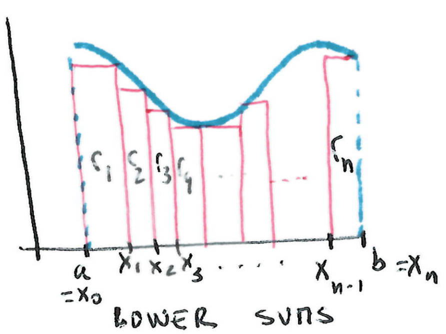
2 Approximations:

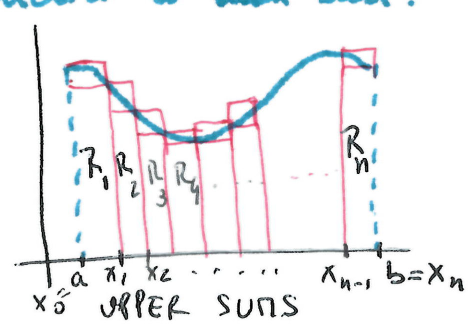
(1) Overestimate (Upper Riemann Sums) → in trap

(2) Underestimate (Lower ——————) → exhaust

Both use rectangles with · base on the x-axis of small length

· height: either a max (for (1)) or min (for (2)) of f restricted to each base.





STEP 1: Subdivide $[a,b]$ into n piece of (the same) small length using $n+1$ points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

Length of each piece: $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \ldots, n-1, n$.

[To simplify, can assume all lengths agree so $\Delta x = \frac{b-a}{n}$]

STEP 2: Build 2 rectangles for each segment $[x_{k-1}, x_k]$:

(1) Upper Rectangle R_k of base $[x_{k-1}, x_k]$ & height $M_k = \max\{f(x) : x_{k-1} \le x \le x_k\}$

(2) Lower —— r_k —— $m_k = \min\{$ —— $\}$.

Note: Want m_k & M_k to be real numbers. If f is cont, this will be true by EVT.

Area$(r_k) = m_k \Delta x_k$ & Area$(R_k) = M_k \Delta x_k$.

STEP 3: Areas cover by small & big rectangles satisfy:

$$A_{(r)} := A(r_1 \cdots r_n) = \sum_{k=1}^{n} \text{Area}(r_k) \le \text{Area}(S) \le \sum_{k=1}^{n} \text{Area}(R_k) = \text{Area}(R_1 \cdots R_n) =: A(R)$$

- As we increase n & we decrease all Δx_k, $A(r)$ grows & $A(R)$ decreases

Thm: If f is continuous, both areas have the same limit as $n \to \infty$, $\Delta x_k \to 0$.
This limit is precisely Area(S)

Why? Since f is cont, we can use EVT to find $\underline{x_k}$, $\overline{x_k}$ in $[x_{k-1}, x_k]$ where $m_k = f(\underline{x_k})$ & $M_k = f(\overline{x_k})$

If we pick any point x_k^* in $[x_{k-1}, x_k]$ we'll have

$$\text{Area}(r_k) = f(\overline{x_k})\Delta x_k \le \underbrace{f(x_k^*)\,\Delta x_k}_{\text{Area of rect with base } [x_{k-1}, x_k] \text{ \& ht} = f(x_k^*)} \le f(\overline{x_k})\Delta x_k = \text{Area}(R_k)$$

Call $\displaystyle\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ = Riemann Sum.

We get $A(r) \le$ Riemann Sum $\le A(R)$.

- If $\delta = \max\limits_{k=1,\dots,n} (\Delta x_k) \to 0$ then **all** $\Delta x_k \to 0$. Furthermore, this forces $n \to \infty$.
- If we show $\boxed{\lim\limits_{\delta \to 0} A(r) = \lim\limits_{\delta \to 0} A(R)}$ (*), then by the Squeeze Theorem we'll have $\text{Area}(S) = \lim\limits_{\delta \to 0} \sum\limits_{k=1}^{n} f(x_k^*)\,\Delta x_k$ for $\delta = \max\limits_{k=1,\dots,n}(\Delta x_k)$.

for **any** choice of pts x_k^* in $[x_{k-1}, x_k]$. $k = 1, \dots, n$

$$\left[\begin{array}{l} \underline{\text{EX}}: \cdot\ x_k^* = x_{k-1} \ (\text{LEFT RS}) \\ \cdot\ x_k^* = x_k \ (\text{RIGHT RS}) \\ \cdot\ x_k^* = \frac{x_k + x_{k-1}}{2}. \end{array}\right]$$

Def We set the definite integral $\int_a^b f(x)\,dx$ as this Area (S), viewed as the limit of Riemann Sums.

Notation: a = lower limit of integration, $f(x)$ = integrand

b = upper ———————, x = variable of integration

- We say f is **integrable** over $[a, b]$ if the limit of the Riemann Sums exists & is independent of **all** our choices of pts $(x_0, \dots, x_n, x_1^*, \dots, x_n^*)$

Q: How to guarantee (*)? **Continuity** of f will show that m_k & M_k will be close enough if Δx_k is sufficiently small. (See Appendix A5)

Next Time: Examples of integrable & non-integrable functions.