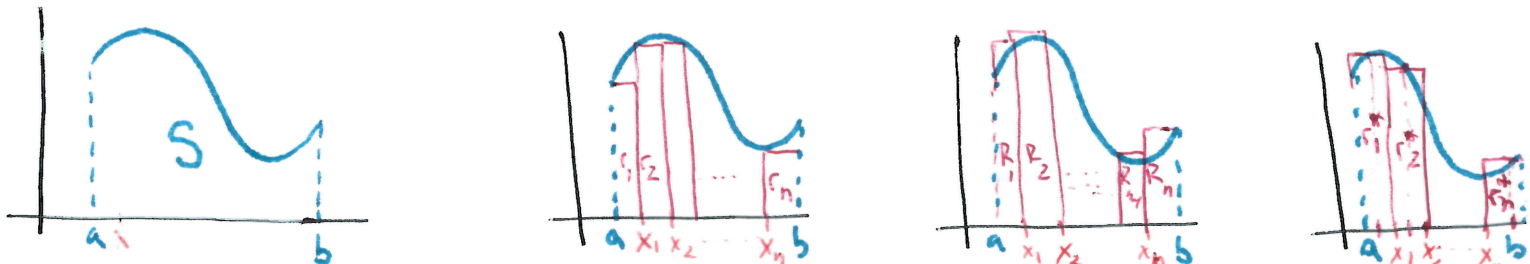
Lecture XXIII §6.5 The computation of areas as limits
§6.7 Properties of definite integrals

§1 Definite integrals

Recall $f:[a,b] \longrightarrow \mathbb{R}_{\geq 0}$ function. What to find the area of the region S under the curve $y=f(x)$ & above the x-axis



$\text{Area}(S) = ?$ LOWER SUM UPPER SUM RIEMANN SUM

STEP 1: Pick $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ & set $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \ldots, n$.

STEP 2: Pick
- $m_k = \min\{f(x) : x_{k-1} \leq x \leq x_k\}$
- $M_k = \max\{f(x) : x_{k-1} \leq x \leq x_k\}$
- $x_k^* =$ any pt in $[x_{k-1}, x_k]$

for $k = 1, \ldots, n$

STEP 3: r_k = Rectangle with base $[x_{k-1}, x_k]$ & ht $= m_k = f(\underline{x_k})$

R_k = ______ ht $= M_k = f(\overline{x_k})$

r_k^* = ______ ht $= f(x_k^*)$

STEP 4: Note $\displaystyle\sum_{k=1}^{n} \text{Area}(r_k) = \sum_{k=1}^{n} m_k \Delta x_k \leq \text{Area}(S) \leq \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k=1}^{n} \text{Area}(R_k)$

Note ______ $\leq \boxed{\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x_k} \leq$ ______

(GENERAL) RIEMANN SUM

Thm If f is continuous, then as $\boxed{\delta = \max \Delta x_k \longrightarrow 0}$, we have $n \to \infty$ &

$$\lim_{\delta\to 0} \sum_{k=1}^{n} \text{Area}(r_k) = \lim_{\delta\to 0} \sum_{k=1}^{n} \text{Area}(R_k).$$

(Proof Appendix A5)

Consequence: By Squeeze Thm, the middle guys have the same limit, so

$$\text{Area}(S) = \lim_{\delta\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \underset{\text{define}}{=} \int_a^b f(x)\,dx. \quad (\text{for } f \text{ cont})$$

We call $\int_a^b f(x)\,dx$ the <u>definite integral</u>

<u>Names</u>: a = lower limit of integration , $f(x)$ = integrand
b = upper ______ , x = variable of integration.

<u>Def</u> We say f is <u>integrable</u> over $[a,b]$ if the limit of the Riemann sums exists and it is independent of all our choices of pts $(x_1,\dots,x_{n-1},x_1^*,\dots,x_n^*)$

<u>Prop</u>: Continuous functions on $[a,b]$ are integrable over $[a,b]$

⚠ Not every function is integrable!

Eg: $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}\end{cases}$ on $[0,1]$ (discontinuous everywhere!)

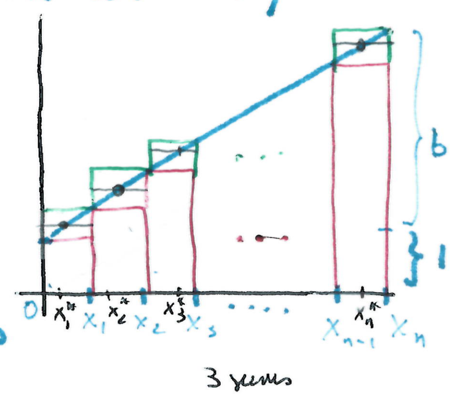
⇝ Lower sums : $m_k = 0$ for all k. so $A(r) = \sum_{k=1}^n 0 \cdot \Delta x_k^* = 0 \cdot \text{len}([0,1]) = 0$

Upper ___ : $M_k = 1$ ______ $A(R) = \sum_{k=1}^n 1\,\Delta x_k^* = 1$.

So f is not integrable (general Riemann sums have no limit)

§2 Examples:

① $f:[0,b] \longrightarrow \mathbb{R}_{\geq 0}$ $f(x) = x+1$.
continuous, so integrable! $\boxed{\text{Area} = b^2/2 + b}$



• Points : equidistant $x_0 = 0,\ x_1 = \frac{b}{n},\ x_2 = \frac{2b}{n}, \dots, x_n = b$

. Lower Sums : $\underline{x}_k = x_k^* = x_{k-1}$ ⇝ $m_k = f(x_{k-1}) = 1 + x_{k-1}$
(= Left Sums)

. Upper Sums $\overline{x}_k = x_k^* = x_k$ ⇝ $M_k = f(x_k) = 1 + x_k$
(= Right Sums)

} because f is increasing

. Arbitrary x_k^* : $f(x_k^*) = 1 + x_k^*$.

$$\sum_{k=1}^n \text{Area}(r_k) = \sum_{k=1}^n m_k\,\Delta x_k = \sum_{k=1}^n (1 + \underbrace{x_{k-1}}_{=(k-1)\frac{b}{n}})\frac{b}{n} = \frac{b}{n}\sum_{k=1}^n \left(1 + (k-1)\frac{b}{n}\right)$$

$$= \frac{b}{n}\sum_{k=1}^n 1 + \frac{b^2}{n^2}\underbrace{\sum_{k=1}^n (k-1)}_{=0+1+\dots+n-1} = \frac{b}{n}n + \frac{b^2}{n^2}\sum_{k=1}^{n-1} k = b + \frac{b^2}{n^2}\frac{n(n-1)}{2}$$

$$= b + \frac{b^2}{2}\left(1 - \frac{1}{n}\right)$$

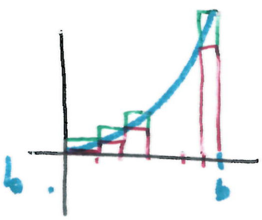
Similarly, $\sum_{k=1}^{n} \text{Area}(R_k) = \sum_{k=1}^{n} (1+\overset{= kb/n}{x_k^*})\frac{b}{n} = \frac{b}{n}\, n + \frac{b^2}{n^2}\sum_{k=1}^{n} k$

$\delta = \frac{b}{n} \to 0$ thus $n \to \infty$ $\quad = b + \frac{b^2}{n^2}\,\frac{n(n+1)}{2} = b + \frac{b^2}{2}\left(1+\frac{1}{n}\right)$

So
$$\left.\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{n} \text{Area}(r_k) &= \lim_{n\to\infty} b + \frac{b^2}{2}\left(1+\frac{1}{n}\right) = b + \frac{b^2}{2} \\ \lim_{n\to\infty} \sum_{k=1}^{n} \text{Area}(R_k) &= \lim_{n\to\infty} b + \frac{b^2}{2}\left(1+\frac{1}{n}\right) = b + \frac{b^2}{2} \end{aligned}\right\} \underline{\text{same}}!$$

So $\int_0^b (1+x)\,dx = b + \frac{b^2}{2} = \lim_{n\to\infty} \sum_{k=1}^{n} (1+x_k^*)\frac{b}{n}$.

(2) $f : [0,b] \longrightarrow \mathbb{R}^2$ $\quad f(x) = x^2$ cont ✓ $\quad \boxed{\text{Area} = ?}$



· Points equidistant $x_0 = 0,\ x_1 = \frac{b}{n},\ x_2 = \frac{2b}{n}, \dots, x_n = b$.

· f increasing so $\underline{x_k} = x_{k-1} = \frac{(k-1)b}{n} \le \overline{x_k} = x_k = k\frac{b}{n}$

Lower sums $= \sum_{k=1}^{n} x_{k-1}^2 \frac{b}{n} = \frac{b}{n}\sum_{k=1}^{n}(k-1)^2\frac{b^2}{n^2} = \frac{b^3}{n^3}\sum_{k=1}^{n}(k-1)^2 = \frac{b^3}{n^3}\sum_{k=1}^{n-1} k^2$ $\quad 0 + 1^2 + 2^2 + \dots + (n-1)^2$

$= \frac{b^3}{n^3}\,\frac{(n-1)(n)(2(n-1)+1)}{6} = \frac{b^3}{6}\,\frac{(n-1)(2n-1)}{n^2} = \frac{b^3}{6}\left(2 - \frac{3}{n} + \frac{1}{n^2}\right)$

Upper sums $= \sum_{k=1}^{n} x_k^2 \frac{b^2}{n^2} = \frac{b^3}{n^3}\sum_{k=1}^{n} k^2 = \frac{b^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$

$= \frac{b^3}{6}\,\frac{(n+1)(2n+1)}{n^2} = \frac{b^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$

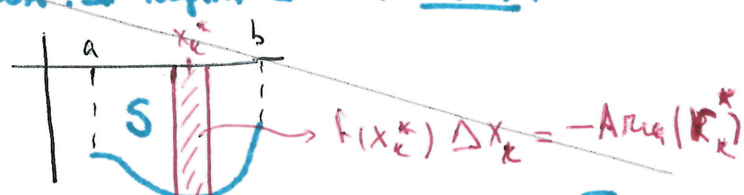
Both sums have the same limit as $n \to \infty$, namely $\frac{b^3}{3}$.

So $\int_0^b x^2\,dx = \boxed{\frac{b^3}{3}} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\frac{b}{n}$.

§3 Algebraic vs geometric area

. Geometric area = f is cont & positive, & region = area under the curve.

·Q: What to do if $f(x) \le 0$?

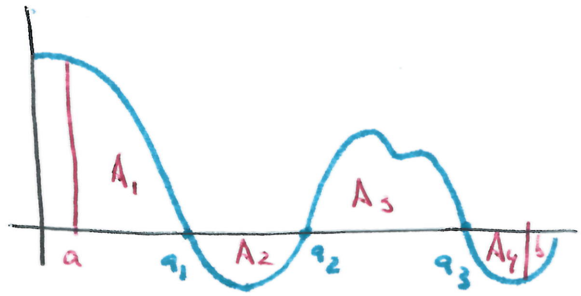


A: $\int_a^b f(x)\,dx = -\text{Area}(S) = -\int_a^b (\underbrace{-f(x)}_{\ge 0})\,dx$. if $f(x) \le 0$ & continuous.

We define the algebraic area (signed area) as $\int_a^b f(x)\,dx$.

In general: If $f(x)$ cont. & goes above & below the x-axis:

- Signed area (algebraic) $= \int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4$
- Geometric area $= \int_a^b |f(x)|\,dx = A_1 + A_2 + A_3 + A_4$



Q: How to compute this? Find the zeros $a_1 < \cdots < a_{n-1}$ of f on $[a,b]$ $\quad a_0 = a$, $a_n = b$

Each area will correspond to f restricted to $[a_{k-1}, a_k]$ $\quad k = 1, \dots, n$

$$A_k = \int_{a_{k-1}}^{a_k} |f(x)|\,dx \quad \& \quad \text{sign}(A_k) = \begin{cases} + & \text{if } f \geq 0 \text{ on } [a_{k-1}, a_k] \\ - & \text{if } f \leq 0 \end{cases}$$

$\quad\quad \| \; \pm f(x)$ depending on the sign of f.

$$\text{Area} = \sum_{k=1}^{n} A_k = \int_a^b |f(x)|\,dx \quad \& \quad \text{Signed area} = \sum_{k=1}^{n} \text{sign}(A_k)\,A_k = \int_a^b f(x)\,dx$$

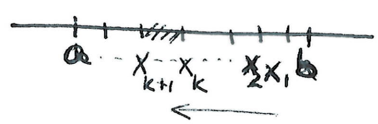
§4 General Properties:

(1) If $a < b$, what is $\int_b^a f(x)\,dx$? $= -\int_a^b f(x)\,dx$.

(if $a = b$, then $\int_a^a f(x)\,dx = 0$)

Reason: $\Delta x_k = x_{k+1} - x_k < 0$

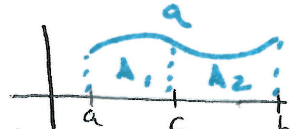
$f(x_k^*)$ is the same for both orders



we move from b to a. So $a = x_n < x_{n-1} < \cdots < x_0 = b$

(2) Additivity I: If $a < c < b$ $\quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

& also true for geometric area / signed area.



TOTAL area $= A_1 + A_2$.

(3) Scalar Mult: $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$ by Limit Laws

Why? $$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \underbrace{(\lambda f)(x_k^*)}_{= \lambda f(x_k^*)} \Delta x_k = \lambda \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

(4) Additivity 2 : $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
if both f & g are continuous on $[a,b]$.

Why? Again by limit laws, using $(f+g)(x_k^*) = f(x_k^*) + g(x_k^*)$

Next Time: How to compute all these definite integrals?

Answer: Fundamental Theorem of Calculus.