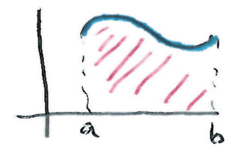
Lecture XXIV §6.7 Algebraic vs Geometric Areas
§6.6 The Fundamental Thm of Calculus

§1 Algebraic vs geometric area

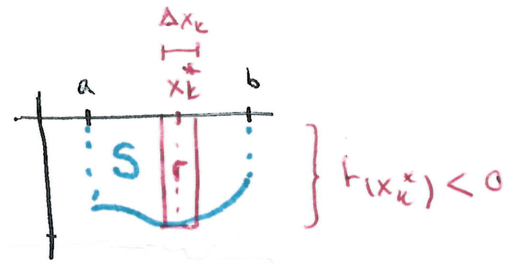
- If f is cont. & positive on $[a,b]$, then



Geometric Area = area under the curve & above x-axis

$$= \int_a^b \underbrace{f(x)}_{\geq 0}\, dx.$$

- Q: What to do if $f(x) \leq 0$?



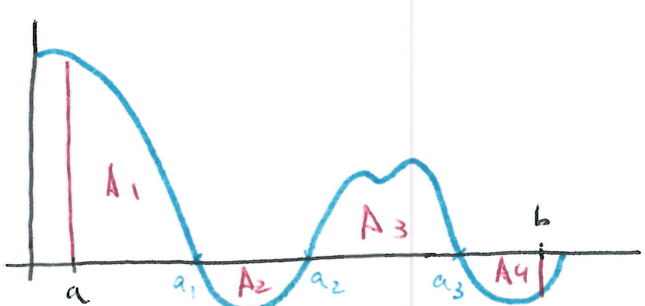
Use Riemann sums with rectangles of height $f(x_k^*)$

$$(f(x_k^*)\,\Delta x_k) = -\text{Area}(\text{rectangle})$$

So $$\int_a^b f(x)\,dx = -\text{Area}(S) = -\int_a^b \underbrace{(-f_{(x)})}_{\geq 0}\,dx = -\int_a^b |f(x)|\,dx$$

- We call $\int_a^b f(x)\,dx$ the <u>algebraic</u> (or signed) <u>area</u>

- <u>In general</u>: If f is cont & goes above & below the x-axis:



<u>STEP 1</u>: Find and order the zeros of f (x-intercepts). $a_1 < \dots < a_{n-1}$.
Set $a_0 = a$ & $a_n = b$.

<u>STEP 2</u>: Each area (bounded by the graph of f on $[a_{k-1}, a_k]$ & the x-axis) has a constant sign, so

we have $A_k = \int_{a_{k-1}}^{a_k} |f(x)|\,dx$ & $\text{sign}_{A_k}(f) := \begin{cases} + & \text{if } f \geq 0 \text{ on } [a_{k-1}, a_k] \\ - & \text{if } f \leq 0 \text{ ———} \end{cases}$

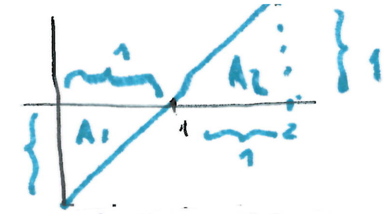
$$\text{Geom Area} = \sum_{i=1}^{n} A_k = \int_a^b |f(x)|\,dx$$

$$\text{Signed Area} = \sum_{i=1}^{n} \text{sign}_{A_k}(f) \cdot A_k = \sum_{i=1}^{n} \int_{a_{k-1}}^{a_k} \underbrace{f(x)}_{\text{sign } f\,|f(x)|}\,dx = \int_a^b f(x)\,dx.$$

(Notice we are NOT using "Additivity Properties" from last lecture.)

<u>Eg.</u> Geom Area $= A_1 + A_2 + A_3 + A_4$ & Alg Area $= A_1 - A_2 + A_3 - A_4$.

Example: $f(x) = x-1$ on $[0,2]$



Zeros of f: only 1 $\quad a_1 = 1$

Signed Area $= A_1 - A_2 = 0$

Geom Area $= A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$

$$\begin{cases} A_1 = \frac{1\cdot 1}{2} = \int_0^1 \underbrace{|x-1|}_{\leq 0} dx = \int_0^1 (1-x)\,dx \\ A_2 = \frac{1\cdot 1}{2} = \int_1^2 \underbrace{|x-1|}_{\geq 0} dx = \int_1^2 (x-1)\,dx \end{cases}$$

CONSEQUENCE Properties for definite integrals of pos & cont functions are true for ALL integrals of all cont functions

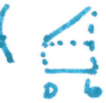
Next Time: View this as the area between 2 curves: the graph of f & the x-axis, viewed as the graph of $g(x)=0$. (§7.1 & 7.2)

§2 The Fundamental Theorem of Calculus

- Fundamental = it relates differential & integral calculus.

Thm: Assume $f:[a,b] \longrightarrow \mathbb{R}$ is cont., and let $F(x)$ be any antiderivative of f (write $F = \int f(x)\,dx$). Then, the signed area between the graph of f and the x-axis between a & b equals:

$$\int_a^b f(x)\,dx = F(b) - F(a) =: F(x)\Big|_a^b .$$

. **Example** (last time) $\quad f(x) = 1+x$. on $[0,b]$, Area () $= b + \frac{b^2}{2}$

$F(x) = x + \frac{x^2}{2}$ is an antiderivative of f, $\quad F(b) = b + \frac{b^2}{2}$, $F(0) = 0$ ✓

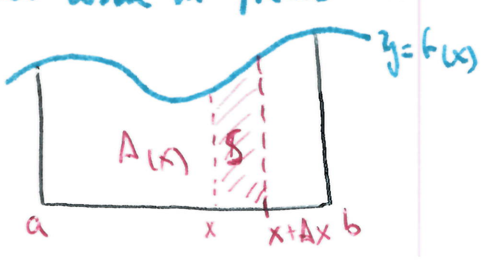
. **Example above**: Sign Area $= 0$ $\quad$; $\quad F(x) = \frac{x^2}{2} - x \qquad F(0) = 0,\ F(2) = 2-2 = 0$

. **Example** $\quad f(x) = x^2$ on $[0,b]$

. Calculation with lower & upper Riemann Sums give Area $= \frac{b^3}{3}$

. $F(x) = \frac{x^3}{3} \qquad F(0) = 0,\ F(b) = \frac{b^3}{3}$.

Proof idea (Newton-Leibniz) For simplicity, we assume $f(x) \geq 0$. Otherwise we work on pieces with constant sign & use additivity (e.g. $\int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
$= \underline{F(c)} - F(a) + F(b) - \underline{F(c)}$
$= F(b) - F(a)$)



Given x in $[a,b]$ we define the (signed) area function $x \longmapsto A(x) = \int_a^x f(t)\,dt$

Note that we've change notation from x to t inside the integral to avoid confusion.

It seems that $A(x)$ is a smooth function whenever f is continuous.
We verify this by computing $A'(x)$ via the method of increments:

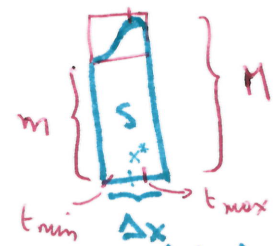
$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x}$$

<u>WANT TO SHOW</u>: $\frac{dA}{dx} = f(x)$.

What is this ratio? It's the $\frac{\text{Area of } S}{\Delta x}$ in the picture!

<u>STEP 1</u>: We want to show area of S is small if $\underset{\text{(base)}}{\Delta x}$ is small.

How? Use max & min estimates!



$0 \leq m = \min\{f(t) : t \text{ in } [x, x+\Delta x]\}$

$0 \leq M = \max\{f(t) : t \text{ in } [x, x+\Delta x]\}$

- Since f is cont, the EVT tells us $m = f(t_{min})$ & $M = f(t_{max})$ for 2 ~~points~~ t_{min}, t_{max} in $[x, x+\Delta x]$.

So $\boxed{m\,\Delta x \leq \text{Area}(S) \leq M\,\Delta x}$

<u>STEP 2</u>: $f(t_{min}) = m \leq \dfrac{\text{Area}(S)}{\Delta x} \leq M = f(t_{max})$.

(*)
- But f is continuous on $[x, x+\Delta x]$, so by the Intermediate Value Theorem, all values between m & M can be achieved. So we can find some x^* in $[x, x+\Delta x]$ with $\boxed{f(x^*) = \dfrac{\text{Area}(S)}{\Delta x}}$.
- The cont of f again says that $f(x^*) \xrightarrow[\Delta x \to 0]{} f(x)$.

(<u>Formally</u>: for any $\varepsilon > 0$, we can find $\delta > 0$ so that if $|x^* - x| < \delta$ then $|f(x^*) - f(x)| < \varepsilon$. Picking $\Delta x < \delta$ ensures $|x^* - x| \leq \Delta x < \delta$ ✓)

<u>Conclusion</u>: $\dfrac{dA}{dt} = \lim_{\Delta x \to 0} f(x^*) = f(x)$.

<u>Alternative to (*)</u>:
- $t_{min}, t_{max} \xrightarrow[\Delta x \to 0]{} x$ & f is cont
so $f(t_{min}), f(t_{max}) \xrightarrow[\Delta x \to 0]{} f(x)$. Squeeze Thm gives $\dfrac{\text{Area}(S)}{\Delta x} \to f(x)$

<u>STEP 3</u>: By definition $A(x) = \int_a^x f(t)\,dt$ is an antiderivative for $f(x)$.

By "uniqueness" we ~~can~~ find a constant C satisfying $A(x) = F(x) + C$ for all x

How to find C? We evaluate at a convenient x!

$$A(a) = \int_a^a f(t)\,dt = 0 \overset{?}{=} F(a) + C \qquad \text{so } C = -F(a).$$

We conclude $A(x) = F(x) - F(a)$ for all x.

Evaluate at $x=b$ to get $A(b) = \int_a^b f(t)\,dt = F(b) - F(a)$. $\square$

Note: Why does the result not depend on our choice of antiderivative?

Any other antideriv differs from F by a constant.

So if $G(x) = F(x) + B$, then $G(b) - G(a) = (F(b) + B) - (F(a) + B)$
$= F(b) - F(a)$.

Note: The proof works no matter the sign of f, just replace $\frac{\text{Area}(S)}{\Delta x}$ by $\frac{\text{signed Area}(S)}{\Delta x}$.

Consequences (1) If f is continuous $\boxed{\left(\int_a^{x} f(t)\,dt\right)' = A'(x) = f(x)}$.

(2) $\left(\int_x^a f(t)\,dt\right)' = \left(-\int_a^x f(t)\,dt\right)' = -\left(\int_a^x f(t)\,dt\right)' = -f(x)$.

(3) If f is cont & $u(x)$ is differentiable:

$$\frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = \frac{d}{dx}\left(A(u(x))\right) \underset{\substack{\downarrow\\ \text{Chain}\\ \text{Rule}}}{=} A'(u(x))\,u'(x) = f(u(x))\cdot u'(x)$$

Example $\left(\int_0^{\sin x} (t+1)\,dt\right)' = (t+1)\big|_{(\sin x)} \cdot (\sin(x))' = (\sin x + 1)\cos(x)$