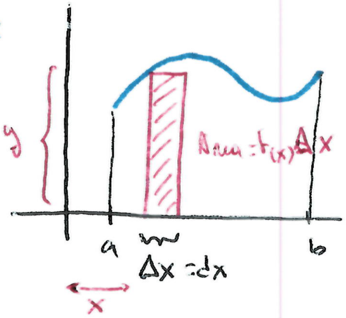
Lecture XXV : §7.1 The intuitive meaning of integration
§7.2 The area between two curves

Recall :



FTC : If $f:[a,b] \longrightarrow \mathbb{R}$ is continuous, then
$A_{(x)} = \int_a^x f_{(t)}\,dt$ is the signed area of the region between the graph of f restricted to $[a,x]$ & the x-axis.
If F is an antiderivative of f, we have

$$A(b) = \int_a^b f_{(t)}\,dt = F_{(b)} - F_{(a)} = F_{(x)}\Big|_a^b$$

Consequence 1
The proof showed $\boxed{A'_{(x)} = f_{(x)}.}$

Example $\left(\int_0^{x} \sin t\,dt\right)' = \sin(x)$ (compare by FTC: $\int_0^x \sin t\,dt = -\cos t\Big|_0^x = -\cos x + 1.$)

Consequence 2 :
- $\left(\int_x^a f_{(t)}\,dt\right)' = \left(-\int_a^x f_{(t)}\,dt\right)' = -\left(\int_a^x f_{(t)}\,dt\right)' = -f_{(x)}.$
- If f is cont & $u_{(x)}$ is diff'ble :

$$\frac{d}{dx}\left(\int_a^{u(x)} f_{(t)}\,dt\right)' = f_{(u_{(x)})}\,u'(x).$$

Why? $(LHS) = \frac{d}{dx}\left(A\left(u_{(x)}\right)\right) \underset{\text{Chain Rule}}{=} A'_{(u_{(x)})}\,u'_{(x)} \underset{\text{FTC}}{=} f_{(u_{(x)})}\,u'_{(x)}$

Example 2 $\int_0^{x^3+\cos x} \sin t\,dt = \sin\left(x^3+\cos x\right)\cdot\left(3x^2 - \sin x\right).$

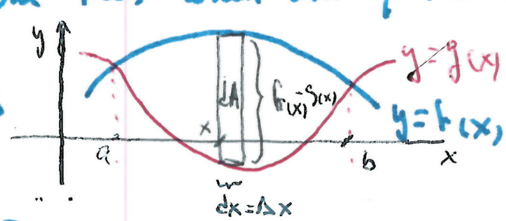
Q: What about differentials? $dA = A'\,dx = f\,dx =$ signed area of the rectangle with base $[x, x+\Delta x]$.
(*) [see last page] ("Element of area")

§1. The area between two curves

GOAL : Compute the area of a region bounded by 2 smooth curves

So far, we've done this when one of the curves was the x-axis.

Simplest examples



f & g satisfy :
(1) $f_{(a)} = g_{(a)}$, $f(b) = g(b)$
(2) For $a < x < b$: $f_{(x)} > g_{(x)}$

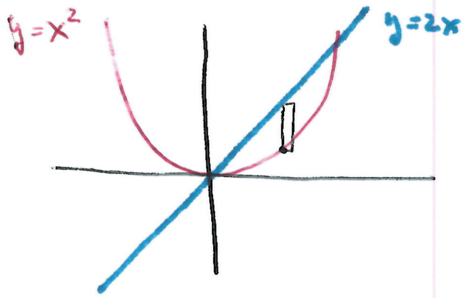
- Height of each strip : $f_{(x)} - g_{(x)} > 0$ ✓

- Length of the base $= dx$
- Area of each rectangle $= (f(x) - g(x))\, dx = dA$

So Area $= \int_a^b f(x) - g(x)\, dx$.

<u>Example 1</u>: $g(x) = x^2$, $f(x) = 2x$. Find the area of the bounded region enclosed by these 2 curves.

<u>STEP 1</u>: Draw the curves & find a, b ⟶ Points where the 2 curves meet.



$g(x) = f(x)$
$x^2 = 2x$ ⟶ $x^2 - 2x = x(x-2) = 0$
so $x = 0$ & $x = 2$

$\boxed{a = 0 \ \ \& \ \ b = 2}$

<u>STEP 2</u>: Check which function is larger on $[a, b]$. Do it by picking any point in between.

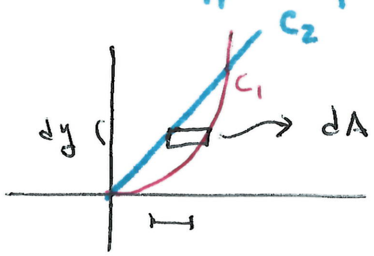
Pick $\frac{a+b}{2} = 1$ $\quad g(1) = 1$ vs $f(1) = 2$ ⟶ $g(x) \leq f(x)$.
$<$

So we conclude $g(x) = x^2 < 2x = f(x)$ on $[0, 2]$.

<u>STEP 3</u>: Use FTC to find the area $A = \int_a^b dA = \int_a^b \underbrace{|f(x) - g(x)|}_{\geq 0 \text{ so drop } |\,|} dx$.

$$A = \int_a^b f(x) - g(x)\, dx = \int_0^2 (2x - x^2)\, dx = x^2 - \frac{x^3}{3} \Big|_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \boxed{\frac{4}{3}}$$

<u>Q</u>: What happens if we use horizontal strips?



⟶ $dA = ?$

<u>STEP 1</u>: Write 2 curves as functions of y & find bounds $= y(a), y(b)$

C_1. $y = f(x) = 2x$ gives $x = \frac{y}{2} = h(y)$

C_2. $y = g(x) = x^2$ & $y > 0$ gives $x = \sqrt{y} = P(y)$

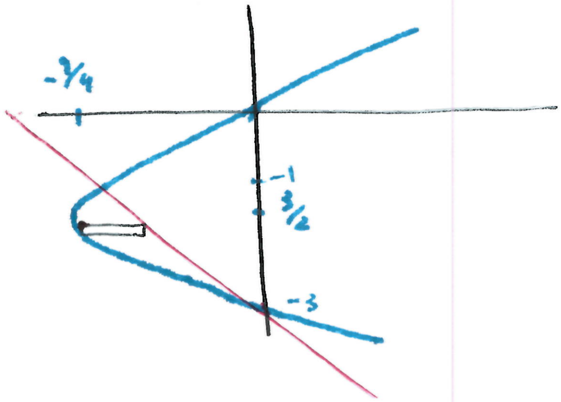
- Bounds: $y(0) = 0$, $y(2) = 4$

<u>STEP 2</u>: Element of area $dA = |h(y) - P(y)|\, dy \underset{\uparrow}{=} (h(y) - P(y))\, dy$

<u>STEP 3</u> Use FTC $\quad A = \int_{y(a)}^{y(b)} |h(y) - P(y)|\, dy \ \underset{\boxed{h \geq P}}{=} \ \int^4 \left(\sqrt{y} - \frac{y}{2}\right) dy$

$$= \frac{2}{3} y^{3/2} - y^2/4 \Big|_0^4 = \left(\frac{2}{3} 4^{3/2} - \frac{16}{4}\right) - 0 = \frac{2\cdot 8}{3} - \frac{16}{4} = \frac{16}{12} = \boxed{\frac{4}{3}}$$

Example 2: Find the area between the curves $x = 3y + y^2$ (parabola) & $x + y + 3 = 0$ (line)



- Vertex of the parabola:
  $\frac{dx}{dy} = 3 + 2y = 0$ so $y = \frac{-3}{2}$ & $x = 3\left(\frac{-3}{2}\right) + \frac{9}{4} = -\frac{9}{4}$
- y-intercepts: $x = 0 = 3y + y^2 = y(3 + y)$
  so $y = 0$ or $y = -3$.
- Line: x-intercepts: $x + 0 + 3 = 0 \Rightarrow x = -3$ & $y = 0$
  y- —— : $0 + y + 3 = 0 \Rightarrow x = 0$ & $y = -3$

Find intersection of 2 curves:

$$\underset{\text{parab}}{3y + y^2} = x = \underset{\text{line}}{-3 - y} \Rightarrow y^2 + 4y + 3 = 0.$$

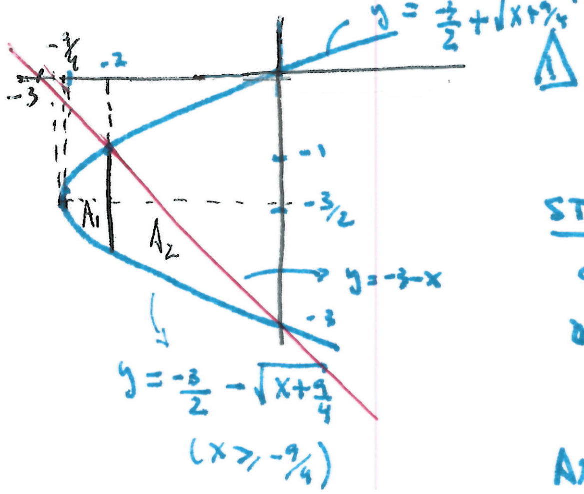
Solutions: $y = -1$ or -3
$(x = -2)$ $(x = 0)$

Seems easier to use horizontal strips $h(y) = 3y + y^2$, $P(y) = -3 - y$ on $[-3, -1]$

- Compare at midpoint: $h(-2) = -6 + 4 = -2$ vs $P(-2) = -3 + 2 = -1$
  $<$
- $A = \int_{-3}^{-1} (P(y) - h(y))\, dy = \int_{-3}^{-1} \underbrace{(-3-y) - (3y + y^2)}_{= -y^2 - 4y - 3}\, dy = \frac{-y^3}{3} - 2y^2 - 3y \Big|_{-3}^{-1}$

$$= \left(\frac{1}{3} - 2 + 3\right) - (9 - 18 + 9) = \boxed{\frac{4}{3}}$$

Q: What if we use vertical strips?



⚠ The upper & lower curves for vertical strips are not given by the same curve!

STEP 1: Divide the picture with vertical lines so that we have the same limiting curves on each vertical strip on the given region

$x = -2$ is used to get A_1 & A_2.

Area $= A_1 + A_2$

STEP 2: Write the limiting curves as functions of $\underline{x}$.

Ignore with y: $y^2+3y-x=0 \rightsquigarrow y = \dfrac{-3 \pm \sqrt{9+4x}}{2}$ $\underline{2 \text{ solutions}}$

$$= -\frac{3}{2} \pm \sqrt{\frac{9}{4}+x}$$

One describes the curve above $y=-\frac{3}{2}$ line & the other below it
(use $+$) (use $-$)

$$A_2 = \int_{-2}^{0} (-3-x) - \left(-\frac{3}{2} - \sqrt{x+\frac{9}{4}}\right) dx$$

$$= \int_{-2}^{0} -\frac{3}{2} - x + \left(x+\frac{9}{4}\right)^{1/2} dx = -\frac{3}{2}x - \frac{x^2}{2} + \frac{2}{3}\left(x+\frac{9}{4}\right)^{3/2} \Big|_{-2}^{0}$$

$$= \frac{2}{3}\left(\frac{9}{4}\right)^{3/2} - \left(3-2+\frac{2}{3}\left(\frac{1}{4}\right)^{3/2}\right) = \frac{2}{3}\left(\frac{3}{2}\right)^3 - \left(1+\frac{2}{3}\frac{1}{8}\right)$$

$$= \frac{9}{4} - \frac{13}{12} = \frac{14}{12} = \frac{7}{6}$$

$$A_2 = \int_{-9/4}^{-2} \left(-\frac{3}{2} + \sqrt{x+\frac{9}{4}}\right) - \left(-\frac{3}{2} - \sqrt{x+\frac{9}{4}}\right) dx = 2\int_{-9/4}^{-2} \sqrt{x+\frac{9}{4}}\, dx$$

$$= \frac{4}{3}\left(x+\frac{9}{4}\right)^{3/2}\Big|_{-9/4}^{-2} = \frac{4}{3}\left(\frac{1}{4}\right)^{3/2} = \frac{4}{3}\frac{1}{8} = \frac{4}{24} = \frac{1}{6}$$

$\underline{\text{TOTAL}}$: $\frac{7}{6} + \frac{1}{6} = \frac{8}{6} = \frac{4}{3}$ (same as before!)

Q $\underline{\text{Multiple crossings}}$? next Time!

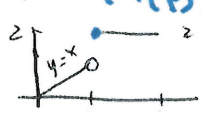
(*) $\underline{\text{Consequence 3}}$: $\left(\displaystyle\int_{v(x)}^{u(x)} f(t)\,dt\right)' = f(u(x))\,u'(x) - f(v(x))\,v'(x)$

$\underline{\text{Why}}$? Add an intermediate pt a $\displaystyle\int_{v(x)}^{u(x)} f(t)\,dt = \int_{v(x)}^{a} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt = -\int_{a}^{v(x)} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt$

So. $\displaystyle\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = \frac{d}{dx}\left(-\int_{a}^{v(x)} f(t)\,dt\right) + \frac{d}{dx}\int_{a}^{u(x)} f(t)\,dt$

$$= -f(v(x))\,v'(x) + f(u(t))\,u'(x) \quad \square$$

Q: What if f is disc. at a pt? $\underline{A}$ use Additivity!



$\displaystyle\int_0^3 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^3 f(t)\,dt$
$= 1/2 + 4$