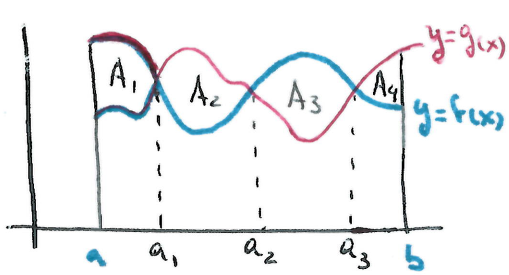
Lecture XXVI: §7.2 The area between two curves
§7.3 Volumes: The Disk Method

§1. The area between two curves with multiple crossings



$f:[a,b]\to\mathbb{R}$ continuous
$g:[a,b]\to\mathbb{R}$

- Geometric Area $= A_1+A_2+A_3+A_4 = \int_a^b |f_{(x)}-g_{(x)}|\,dx$
- Signed Area $= -A_1-A_2+A_3-A_4 = \int_a^b (f_{(x)}-g_{(x)})\,dx$

Q: How to compute A_1, A_2, A_3 & A_4?

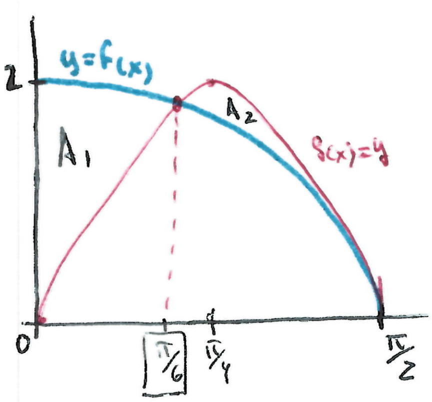
STEP 1: Find all crossings of the 2 curves between $x=a$ & $x=b$
($f(x)=g(x)$ gives $x=a_1,a_2,a_3$)

STEP 2: $A_1 = \int_a^{a_1} (\underbrace{g(x)-f(x)}_{\geq 0})\,dx$, $A_2 = \int_{a_1}^{a_2}(\underbrace{g(x)-f(x)}_{\geq 0})\,dx = \int_{a_1}^{a_2} |f(x)-g(x)|\,dx$

$A_3 = \int_{a_2}^{a_3} (\underbrace{f(x)-g(x)}_{\geq 0})\,dx$, $A_4 = \int_{a_3}^{b} (\underbrace{g(x)-f(x)}_{\geq 0})\,dx$

Areas between consecutive crossings &/or endpoints ⤳ compare $g(x)$ vs $f(x)$ & integrate.

Example: Find the area between $y = \underset{f(x)}{2\cos x}$ & $y = \underset{g(x)}{2\sin(2x)}$ on $[0,\pi/2]$



$f(x) = g(x)$
$2\cos x = 2\sin 2x$
$\cos x = \sin 2x$
$\cos x = 2\sin x\cos x$ ⤳ $\cos x(1-2\sin x)=0$

gives $\cos x = 0$ or $\frac{1}{2} = \sin x$ and $0\leq x\leq \frac{\pi}{2}$

$\boxed{x=\pi/2}$ $\boxed{x=\frac{\pi}{6}}$

Area $= A_1 + A_2$

Compare 2 functions:
- $f(0) = 2\cos(0) = 2$ vs $g(0) = 2\sin 0 = 0$. $>$
- $f(\frac{\pi}{4}) = 2\cos(\frac{\pi}{4}) = 2\frac{\sqrt{2}}{2} = \sqrt{2}$ vs $g(\frac{\pi}{4}) = 2\sin\frac{\pi}{2} = 2$ $<$

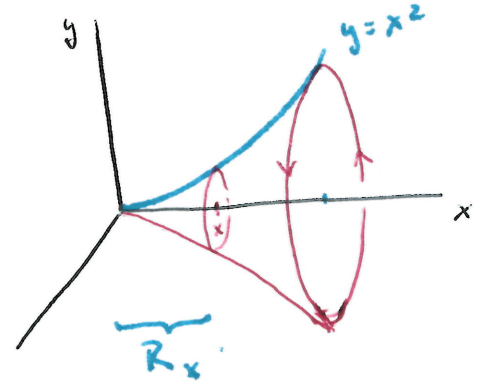
$A_1 = \int_0^{\pi/6} (2\cos x - 2\sin 2x)\,dx = 2\left(+\sin x + \frac{\cos 2x}{2}\right)\Big|_0^{\pi/6} = 2\cdot\frac{1}{2}+\frac{1}{2} - (0+1) = \boxed{\frac{1}{2}}$

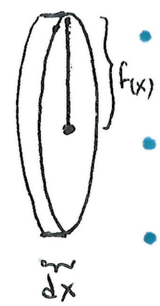
$A_2 = \int_{\pi/6}^{\pi/2} -(2\cos x - 2\sin 2x)\,dx = -(2\sin x + \cos 2x)\Big|_{\pi/6}^{\pi/2} = -(2-1)+(2\tfrac{1}{2}+\tfrac{1}{2}) = \boxed{\frac{1}{2}}$

- TOTAL $= A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = \boxed{1}$
- Signed area $= A_1 - A_2 = \frac{1}{2} - \frac{1}{2} = \boxed{0}$.

§2. Solid of revolution

- Input : Positive & continuous function f on $[a,b]$. **Example** : $f(x) = x^2$, $a=0$, $b=1$
- Rotate the graph about the x-axis to get a solid of revolution R
- Q : What's the volume of R?





- Cross section at x : disk of radius $f(x)$
- Area of the section $= \pi \, (f(x))^2$
- Element of Volume $=$ Area of cross section $\cdot \, dx$

$$\rightsquigarrow dV := \pi (f(x))^2 dx$$

We cover the solid with these vertical slices & let $dx \to 0$.

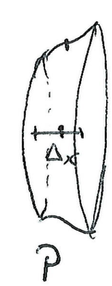
$$\boxed{\mathrm{Vol}(R) = \int_a^b dV = \int_a^b \pi (f(x))^2 dx} \quad (*)$$

Example $f(x) = x^2$ & $[0,1]$. $\mathrm{Vol} = \int_0^1 \pi x^4 dx = \pi \frac{x^5}{5} \Big|_0^1 = \frac{\pi}{5}$

Q: Why does formula (*) work?

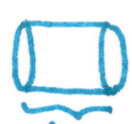
$\mathrm{Vol}(R_x) =$ Volume for solid of revolution between a & x

so $$\mathrm{Vol}(R_x)' = \lim_{\Delta x \to 0} \frac{\mathrm{Vol}(R_{x+\Delta x}) - \mathrm{Vol}(R_x)}{\Delta x}$$



Numerator $= \mathrm{Vol}(P)$

We enclose $\mathrm{Vol}(P)$ between 2 cylinders



with radius r_{min} & r_{max}

with $r_{min} = \min\{f(t) : x \le t \le x+\Delta x\} = f(t_{min})$

$r_{max} = \max\{ \_\_\_\_\_\_\_\_\_\_\_\_ \} = f(t_{max})$

So $\mathrm{Vol}(Cyl_{r_{min}}) \le \mathrm{Vol}(P) \le \mathrm{Vol}(Cyl_{r_{max}})$

$\Delta x \, f(t_{min})^2 \pi \le \mathrm{Vol}(P) \le \Delta x \, f(t_{max})^2 \pi$

So $\pi f^2(t_{min}) \le \dfrac{Vol(P)}{\Delta x} \le \pi f^2(t_{max})$ because $t_{min}, t_{max} \xrightarrow[\Delta x \to 0]{} x$ & f is continuous

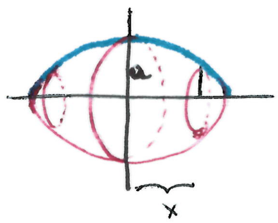
$\pi f^2(t_{min}) \xrightarrow{\Delta x \to 0} \pi f^2(x)$, $\quad \pi f^2(t_{max}) \xrightarrow{\Delta x \to 0} \pi f^2(x)$

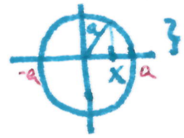
Squeeze Thm gives $Vol(R_x)' = \lim\limits_{\Delta x \to 0} \dfrac{Vol(P)}{\Delta x} = \pi f^2(x)$.

FTC gives $Vol(R_x) = \int_a^x \pi f^2(t)\,dt$. for all $a \le x \le b$.

Examples

① **Sphere of radius a** = rotate a half-circle of radius a



. Function : 

$f(x) = \sqrt{a^2 - x^2}$

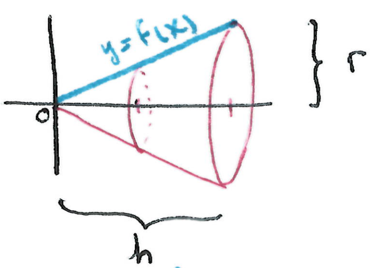
. Endpoints = $-a$ & a

$$dV = \pi f(x)^2\,dx = \pi (a^2 - x^2)\,dx$$

$$Vol = \int_{-a}^{a} \pi (a^2 - x^2)\,dx = \pi \left(a^2 x - \frac{x^3}{3}\right)\Big|_{-a}^{a} = \pi \left(a^3 - \frac{a^3}{3} - \left(-a^3 + \frac{a^3}{3}\right)\right)$$

$$= 2\pi\left(a^3 - \frac{a^3}{3}\right) = \boxed{\frac{4}{3}\pi a^3}$$

② **Cone of height = h & radius of base = r** = rotate a line segment.



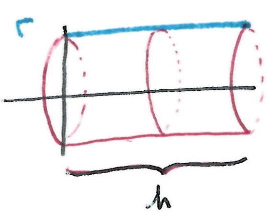
Function: $y = mx$ $\quad y(h) = r$ so $f(x) = \frac{r}{h} x$

Endpoint: 0 & h

$$dV = \pi f(x)^2\,dx = \pi \frac{r^2}{h^2} x^2\,dx$$

$$Vol = \int_0^h \pi \frac{r^2}{h^2} x^2\,dx = \pi \frac{r^2}{h^2} \frac{x^3}{3}\Big|_0^h = \boxed{\frac{\pi r^2 h}{3}}$$

③ **Cylinder of radius r & height h**: = rotate a line segment



. Function $f(x) = r$

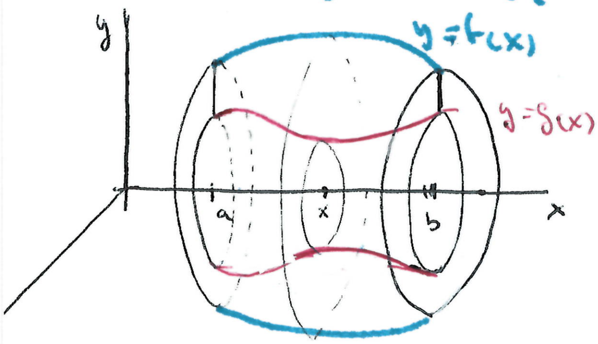
. Endpoints = 0 & h

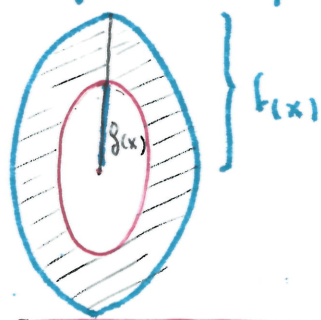
$dV = \pi r^2\,dx \rightsquigarrow Vol = \int_0^h \pi r^2\,dx = \boxed{\pi r^2 h}$

Note: $V_{cone} = \frac{1}{3} V_{cyl}$, $\quad \underset{2r = h}{V_{sphere}} = \frac{2}{3} V_{cyl}$.

§3 Washer Method:

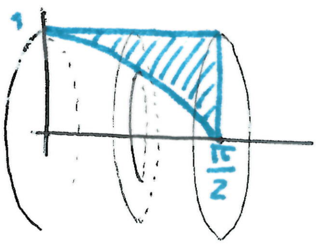
- Input: 2 positive continuous functions f, g on $[a,b]$ with $g(x) \le f(x)$
- Rotate both graphs & get a surface of revolution R
- Cross section = washer = difference of 2 disks of radius $g(x) \le f(x)$





$$dV = \pi\left(f^2(x) - g^2(x)\right)dx \quad \rightsquigarrow \quad \boxed{Vol = \int_a^b dV = \int_a^b \pi\left(f^2(x) - g^2(x)\right)dx}$$

Example: $f(x) = 1$ $\quad g(x) = \sqrt{\cos x} \quad 0 \le x \le \frac{\pi}{2}$



$$dV = \pi(1 - \cos x)\,dx$$

$$Vol = \int_0^{\pi/2} \pi(1-\cos x)\,dx = \pi(x - \sin x)\Big|_0^{\pi/2} = \pi\left(\left(\frac{\pi}{2} - 1\right) - 0\right)$$

$$= \frac{\pi(\pi - 2)}{2}.$$