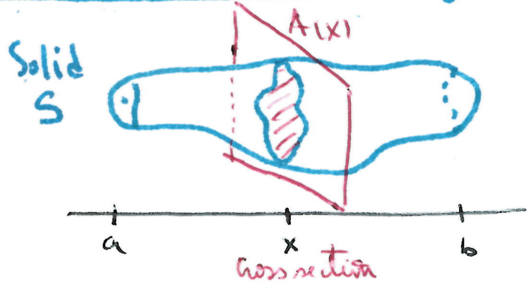
Lecture XXVII: §7.3: Volume via moving slices
§7.4: Volumes: The method of cylindrical shells

§1 Volume via moving slices:

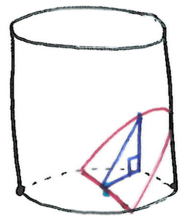


- cross sections : Area = $A(x)$ (x-slices)
- $dV = A(x)\,dx$ & $\text{Vol} = \int_a^b dV = \int_a^b A(x)\,dx$

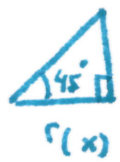
Why? Same idea as of solids of revol. : $\text{Vol}(S_x)' = \lim_{\Delta x \to 0} \dfrac{S_{x+\Delta x} - S_x}{\Delta x} = A(x)$

S_x = solid between a & x.

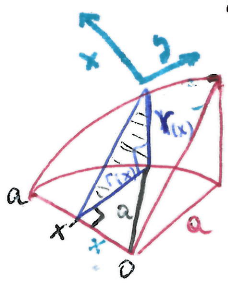
Example:



- 45° plane cuts a cylinder through the center of its base
- radius = a
- Cross section x-axis



$h(x) = r(x)$ = right isosceles Δ



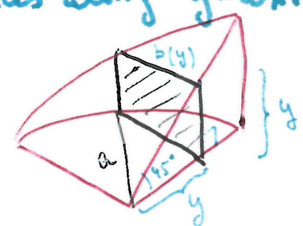
$r(x) = \sqrt{a^2 - x^2}$

$0 \leq x \leq a$

$\text{Vol} = 2\,\text{Vol}($  $) = 2\int_0^a \dfrac{r(x)^2}{2}\,dx = \int_0^a (a^2 - x^2)\,dx = a^2x - \dfrac{x^3}{3}\Big|_0^a$

$= \boxed{\dfrac{2}{3}a^3}$

- Slices along y-axis = rectangles



$\begin{cases} \text{height} = y \\ \text{base} = b(y) = \sqrt{a^2 - y^2} \end{cases}$

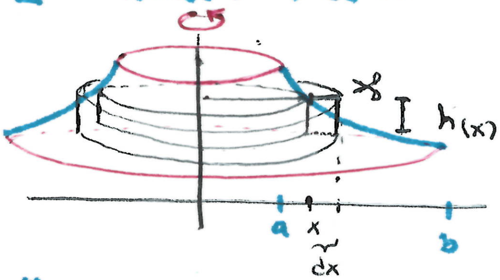
$dV = y\sqrt{a^2 - y^2}\,dy$

$\text{Vol} = 2\int_0^a y\sqrt{a^2 - y^2}\,dy = -\int_{a^2}^0 \sqrt{u}\,du = \int_0^{a^2} \sqrt{u}\,du = \dfrac{2}{3}u^{3/2}\Big|_0^{a^2} = \boxed{\dfrac{2}{3}a^3}$

$u = a^2 - y^2$
$du = -2y\,dy$
$y = 0 \rightsquigarrow u = a^2$
$y = a \rightsquigarrow u = 0$

§ 2 Cylinder shells

- Input: ~~Positive~~ Decreasing & Continuous functions f on $[a,b]$ & $a \geq 0$,
- Rotate the graph about the y-axis to get a solid of revolution R
- Q : What's the volume of R?



$f(b) = m$ = min

$f(a) = M$ = max

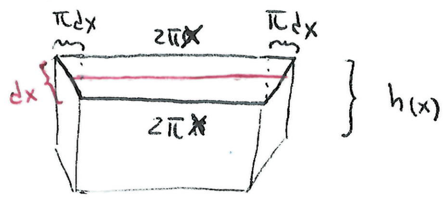
$h(x) = f(x) - m$

Method : Cylindrical shells = $Cyl_1 - Cyl_2$

Cyl_1 = Volume of cylinder of height $h(x)$ & radius $= x + dx$

Cyl_2 = ______________ $h(x)$ & ______ $= x$

Unfold shell :

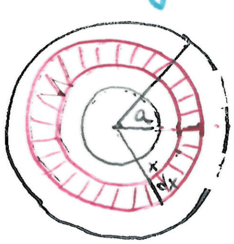


Area $\approx 2\pi x \, dx$

So $dV = 2\pi x \, h(x) \, dx$ $\rightsquigarrow$ $\boxed{\mathrm{Vol} = \int_a^b dV = \int_a^b 2\pi x (f(x) - m) \, dx}$

$[h(x) = f(x) - m]$

- Cross section on $y = m$ plane is a circle of radius b.

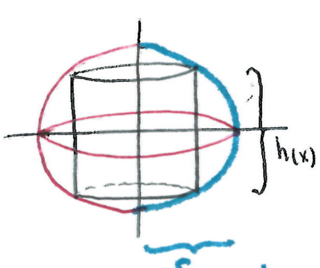


It's covered with concentric circles with radii increased by dx between $r = a$ & $r = b$.

$\begin{cases} \text{Outer circumf} = 2\pi(x + dx) \\ \text{Inner circumf} = 2\pi x \end{cases}$

Examples

① Sphere of radius r $(= 2 \,\mathrm{Vol}(\text{half-sphere}))$



$a = 0$

$b = r$

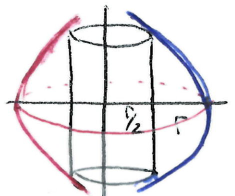
$\left.\begin{array}{ll} f_1(x) = \sqrt{r^2 - x^2} & \text{(top) curve} \\ f_2(x) = -\sqrt{r^2 - x^2} & \text{(bottom)} \end{array}\right\}$ $h(x) = 2\sqrt{r^2 - x^2}$ $(= f_1 - f_2)$

$dV = 2\pi x \, h(x) \, dx = 4\pi x \sqrt{r^2 - x^2} \, dx$

$$V = \int_0^r 4\pi x\sqrt{r^2-x^2}\,dx = \int_{r^2}^{0} -2\pi\sqrt{u}\,du = 2\pi\int_0^{r^2}\sqrt{u}\,du = 2\pi\,\frac{2}{3}u^{3/2}\Big|_0^{r^2} = \boxed{\frac{4}{3}\pi r^3}$$

$u = r^2 - x^2$
$du = -2x\,dx$
$x = 0 \leadsto u = r^2$
$x = r \leadsto u = 0$

1bis Sphere of radius r with a cylinder of radius $\frac{r}{2}$ cut through.



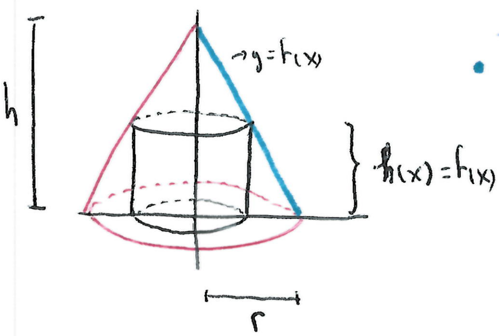
Same dV

Different endpoints: $\frac{r}{2}$ & r

$$\text{Vol} = \int_{r/2}^{r} 4\pi x\sqrt{r^2-x^2}\,dx = 2\pi\,\frac{2}{3}u^{3/2}\Big|_0^{\frac{3}{4}r^2} = \frac{4}{3}\pi\left(\frac{\sqrt{3}}{4}r^2\right)^{3/2} = \boxed{\frac{\sqrt{3}\,\pi r^3}{2}}$$

$x = r \leadsto u = 0$
$x = \frac{r}{2} \leadsto u = \frac{3}{4}r^2$

2 Cone of height h & radius of base r:



- Function $f(0) = h$, $f(r) = 0$

$f(x) = mx + p \leadsto f(0) = p = h$ & $f(r) = mr + h = 0$

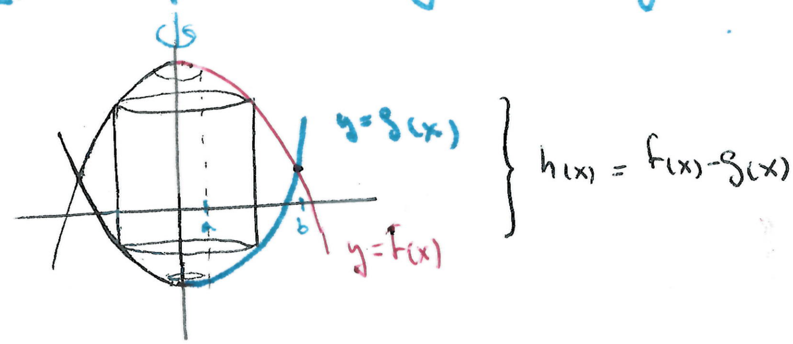
so $m = -\frac{h}{r}$

$$\boxed{f(x) = -\frac{h}{r}x + h}$$ see ✓

- End points $= 0$ & r, $dV = 2\pi x\, f(x)\,dx$

$$\text{Vol} = \int_0^r 2\pi x\left(h - \frac{h}{r}x\right)dx = h2\pi\int_0^r x\left(1-\frac{x}{r}\right)dx = 2\pi h\left(\frac{x^2}{2} - \frac{x^3}{3r}\right)\Big|_0^r = \boxed{\frac{\pi h r^2}{3}}$$

Q What if 2 bounding curves $y = f(x)$ & $y = g(x)$, f, g cut, $f \geq g$?



Height $= f(x) - g(x)$

Endpoints: a (typically 0) & b = meeting pt of $f(x) = g(x)$.

$$dV = 2\pi x\,(f(x) - g(x))\,dx$$

$$\boxed{\text{Vol} = \int_a^b 2\pi x\,(f(x) - g(x))\,dx}$$

Example: $\begin{cases} g(x) = x^2 \\ f(x) = 2 - x^2 \end{cases}$

- Meeting pt $=$ $g(x) = f(x)$

$x^2 = 2 - x^2$

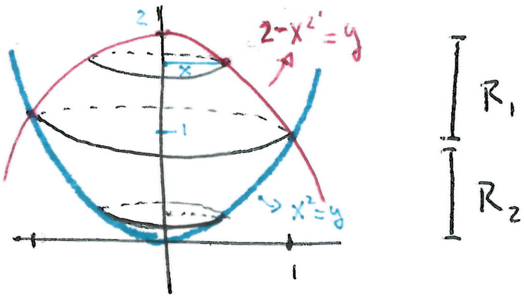
$2x^2 = 2 \leadsto x = \pm 1$ so $\boxed{b = 1}$

- $a = 0$

$$dV = 2\pi x(-x^2+(2-x^2))\,dx = -4\pi x(x^2-1)\,dx$$

$$\text{Vol} = \int_0^1 -4\pi(x^3-x)\,dx = -4\pi\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Big|_0^1 = -4\pi\left(\frac{1}{4}-\frac{1}{2}\right) = \boxed{\pi}$$

Alternative: disk method around y-axis ⇝ Need to view curves as functions of y



- $2-x^2=y$ ⇝ $\sqrt{2-y}=x$
- $\text{Vol} = \text{Vol}(R_1) + \text{Vol}(R_2)$
- $\text{Vol}(R_1)$ disks $dV_1 = \pi x^2 dy = \pi(\sqrt{2-y})^2 dy = \pi(2-y)\,dy$
- $\text{Vol}(R_2)$ disks $dV_2 = \pi x^2 dy = \pi y\,dy$

End pts for R_1 : 1 & 2 ; End pts for R_2 = 0 & 1.

$$\text{Vol}(R_1) = \int_1^2 \pi(2-y)\,dy = \pi\left(2y-\frac{y^2}{2}\right)\Big|_1^2 = \pi\left((4-2)-\left(2-\frac{1}{2}\right)\right) = \frac{\pi}{2}$$

$$\text{Vol}(R_2) = \int_0^1 \pi y\,dy = \pi\frac{y^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

TOTAL : $\frac{\pi}{2}+\frac{\pi}{2} = \boxed{\pi}$