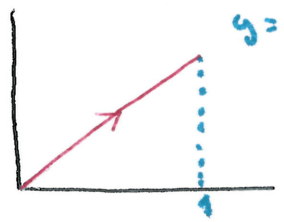
Lecture XXVIII §7.5 Arc Length
§7.6 The area of a surface of revolution

§1 Arc Length

GOAL: Given a curve in the plane, want to compute its length (= length of a string placed on top of it)

Ex 1:

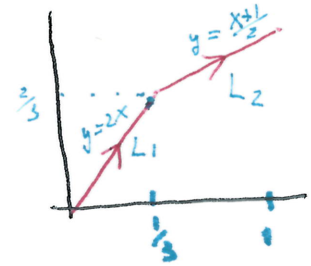


$L = \sqrt{2}$

$(y' = 1)$

$L = \int_0^1 \sqrt{1+1^2}\, dx = \sqrt{2} \cdot 1$

Ex 2



$(y_1' = 2,\ y_2' = \frac{1}{2})$

$L = L_1 + L_2 = \frac{2\sqrt{5}}{3}$

$L_1 = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$

$\shortparallel$

$L_2 = \_\_\_\_\_\_$

$\begin{cases} L_1 = \int_0^{1/3} \sqrt{1+2^2}\, dx = \frac{\sqrt{5}}{3} \\ L_2 = \int_{1/3}^{1} \sqrt{1+\left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}\,\frac{2}{3} \end{cases}$

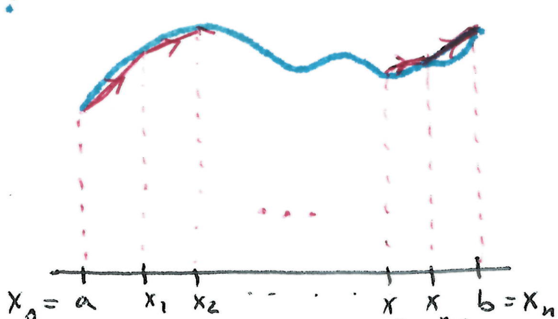
- Length of a polygonal curve = sum of lengths of all chords
- Q: What about general curves?

  A: Approximate them by polygonals and do limit process

Thm: The arc length of the graph $y = f(x)$ of a cont, diff'ble function $f:[a,b] \longrightarrow \mathbb{R}$ whose derivative is continuous is $L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$

[See earlier examples for a numerical verification]

Proof:



STEP 1: Subdivide the interval $[a,b]$ into n intervals $[x_{k-1}, x_k]$ of length

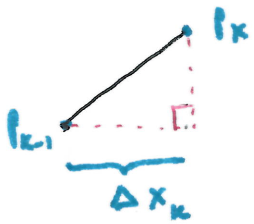
$\Delta x_k = x_k - x_{k-1}$ for $k = 1, \ldots, n$

(can take $\Delta x_k = \frac{b-a}{n}$ for all k)

Call $a = x_0$, $b = x_n$.

STEP 2: Draw the polygonal joining $f(x_{k-1})$ & $f(x_k)$ for $k = 1, \ldots, n$

STEP 3: The segment between $P_{k-1} = (x_{k-1}, f(x_{k-1}))$ & $P_k = (x_k, f(x_k))$ has length



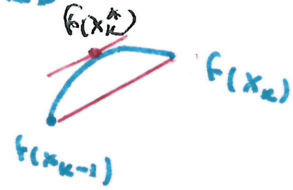
$\} f(x_k) - f(x_{k-1})$

$$L_k = \sqrt{\Delta x_k^2 + (f(x_k) - f(x_{k-1}))^2}$$

$$L_k = \Delta x_k \sqrt{1 + \left(\frac{f(x_k) - f(x_{k-1})}{\Delta x_k}\right)^2}$$

By Mean Value Thm, we can find x_k^* in $[x_{k-1}, x_k]$ with

$$\frac{f(x_k) - f(x_{k-1})}{\Delta x_k} = f'(x_k^*)$$



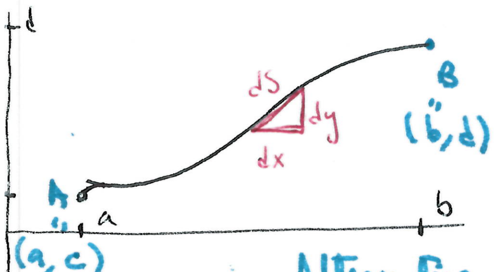
STEP 4: $L(\text{polygonal}) = \sum_{k=1}^{n} L_i = \sum_{k=1}^{n} \Delta x_k \sqrt{1 + f'(x_k^*)^2}$ (Riemann Sum)

$\xrightarrow{\Delta x \to 0} f'(x_k)^2$ (f' cont)

So $L = \lim_{\max(\Delta x_k) \to 0} L(\text{polyg}) = \int_a^b \sqrt{1 + f'(x)^2}\, dx$

NOTE: Often, it is very hard to find the antiderivative of $\sqrt{1+f'(x)^2}$. If so, numerical approximations (with Riemann Sums) are used to find L.

Arc Length element



$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1+(y')^2}\, dx$$

$$= dy\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+(x')^2}\, dy$$

$$L_{AB} = \int_a^b ds = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Alternative $= \int_c^d ds = \int_c^d \sqrt{1 + x'(y)^2}\, dx$. if $y = y(x)$ can be "inverted" to $x = x(y)$.

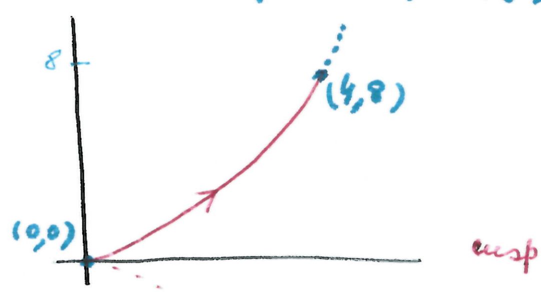
Example 1: Find the length of the curve $y^2 = x^3$ between the points $(0,0)$ & $(4,8)$

Soln 1: Use implicit differentiation

$$2yy' = 3x^2$$

For $y \neq 0$ we get $y' = \frac{3}{2}\frac{x^2}{y}$ cont for $y \neq 0$ (1 bad pt is OK)

$$(y')^2 = \frac{9}{4}\frac{x^4}{y^2} = \frac{9}{4}x.$$



$$L = \int_0^4 \sqrt{1+y'^2}\, dx = \int_0^4 \sqrt{1 + \tfrac{9}{4}x}\, dx = \frac{4}{9}\int_1^{10} \sqrt{u}\, du = \frac{4}{9}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10} = \frac{8}{27}\left(10^{3/2} - 1\right)$$

$u = 1 + \frac{9}{4}x$

Soln 2: Solve for y $\quad y = \sqrt[2]{x^3} \rightsquigarrow y' = \frac{3}{2}\sqrt{x}$ so $y' = \frac{9}{4}x$.

Solve for x $\quad x = y^{2/3} \rightsquigarrow x' = \frac{2}{3}y^{-1/3}$ so $(x')^2 = \frac{4}{9}y^{-2/3}$.

$L = \int_0^8 \sqrt{1 + \frac{4}{9}y^{-2/3}}\, dy$ this is harder to integrate but it can be done!

Note: $\sqrt{1+\frac{4}{9}y^{-2/3}} = \frac{\sqrt{y^{2/3}+\frac{4}{9}}}{y^{1/3}} = y^{-1/3}\sqrt{\frac{4}{9}+y^{2/3}}$

$u = \frac{4}{9}+y^{2/3}$ $\quad du = \frac{2}{3}y^{-1/3}dy$ so $L = \int_{4/9}^{40/9} \sqrt{u}\,\frac{3}{2}du = \frac{8}{27}(\sqrt{10}\,10-1)$

$y=0 \rightsquigarrow u = 4/9$

$y=8 \rightsquigarrow u = \frac{4}{9}+4 = \frac{40}{9}$

Example 2: Circumference of a circle of radius r.

$f(x) = \sqrt{r^2-x^2}$ $\quad -r\leq x\leq r$. $\rightsquigarrow f' = \frac{1}{2}\frac{-2x}{\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}}$

Top half circle

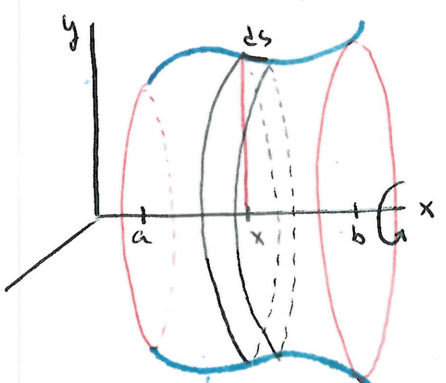
$(f')^2 = \frac{x^2}{r^2-x^2}$

$L = 2\int_{-r}^{r}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 2\int_{-r}^{r}\frac{r}{\sqrt{r^2-x^2}}dx = 2\int_{-r}^{r}\frac{r}{r\sqrt{1-\left(\frac{x}{r}\right)^2}}dr$

$\underset{\substack{u=\frac{x}{r}\\ du=\frac{1}{r}dx}}{=} 2\int_{-1}^{1}\frac{r}{\sqrt{1-u^2}}du = r\underbrace{\left(2\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}du\right)}_{\text{circumference of unit disk}}.$

Later: Use trig substitutions to solve this integral.

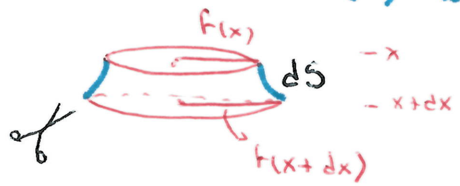
§2 Surfaces of revolution



- Input: $f:[a,b]\to\mathbb{R}$ continuous & positive
- Process: Revolve $y=f(x)$ about the x-axis & take the surface S of the solid of revolution. Call it a surface of revolution

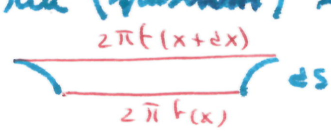
Q: What's Area(S)?

STEP 1: Take n strips $[x_{k-1}; x_k]$ for $k=1,\dots,n$ & take the surface of revolution of f restricted to $[x_{k-1}, x_k]$. $\rightsquigarrow$ Frustum (of a cone)



$\text{Area (frustum)} \overset{(*)}{\approx} 2\pi f(x)\,dS = 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$

Need f' to be cont.



$= 2\pi\left(\frac{f(x)+f(x+dx)}{2}\right)dS$

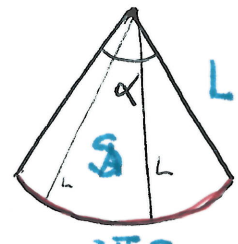
STEP 2: $\lim\limits_{\max \Delta x_k \to 0} \sum\limits_{k=1}^{n}$ Area Frustum $[x_{k-1}, x_k]$ $=$ Area(S)

gives
$$\boxed{\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx = \text{Area}(S) \quad \text{if } f' \text{ is cont.}}$$

Q: Why (*)?

• Model example: cone

Cut open & lay out



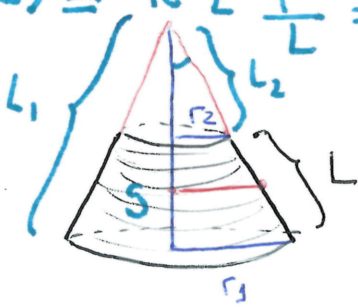
= circular sector of disk of radius L (all pts on the arc are at distance L from the vertex!)

Total area $= \pi L^2 \rightsquigarrow$ sector area $= \pi L^2 \left(\frac{\alpha}{2\pi}\right)$

Circumference of sector $= 2\pi L \boxed{\frac{\alpha}{2\pi}} = 2\pi r \implies \frac{\alpha}{2\pi} = \frac{r}{L}$.

Conclusion: Area$(S) = \pi L^2 \frac{r}{L} = \pi r L$

• Area of Frustum



Start from L, r_1, r_2 & complete to get a cone of side length L_1, with the base of radius r_1.

$$\text{Area } S = \text{Area Cone}(L_1, r_1) - \text{Area Cone}(L_2, r_2)$$
$$= \pi r_1 L_1 - \pi r_2 L_2$$
$$= \pi (r_1 L_1 - r_2 L_2)$$

Similarity of Δ gives $\frac{L_1}{r_1} = \frac{L_2}{r_2}$ so $\boxed{r_2 L_1 = r_1 L_2}$

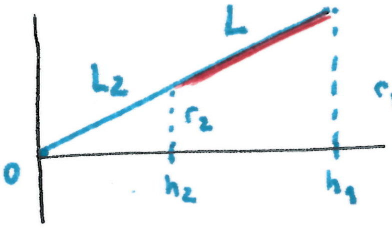
Add and substract $r_1 L_2$ to get:

$$\text{Area}(S) = \pi\Big(r_1 L_1 - r_1 L_2 + \underbrace{\boxed{r_1 L_2}}_{= r_2 L_1} - r_2 L_2\Big) \overset{(*)}{=} \pi\big(r_1(L_1 - L_2) + r_2(L_1 - L_2)\big)$$
$$= \pi (r_1 + r_2)(L_1 - L_2) = \boxed{\pi (r_1 + r_2) L}$$

Conclusion: Area(Frustum) $= 2\pi \left(\frac{r_1 + r_2}{2}\right) L$ → midpoint of 2 radii

⇝ This confirms our formulas from earlier!

Verify this area with our integral formula:



$\rightsquigarrow f(x) = \frac{r_1}{h_1} x \qquad$ so $\quad f'(x) = \frac{r_1}{h_1}$

$$\text{Area(Frustum)} = \int_{h_2}^{h_1} 2\pi \frac{r_1}{h_1} x \sqrt{1 + \frac{r_1^2}{h_1^2}}\, dx$$

$$= \pi \frac{r_1}{h_1^2} \sqrt{h_1^2 + r_1^2}\; x^2 \Big|_{h_2}^{h_1}$$

Want to show: $\pi (r_1 + r_2) L = \pi \frac{r_1 \sqrt{h_1^2 + r_1^2}}{h_1^2} (h_1^2 - h_2^2)$

Use geometry of similar Δ: $\quad \frac{h_2}{h_1} = \frac{L_2}{L_2 + L} = \frac{r_2}{r_1}$, $\sqrt{h_1^2 + r_1^2} = L_2 + L$

So $\frac{\pi r_1 \sqrt{h_1^2 + r_1^2}}{h_1^2} (h_1^2 - h_2^2) = \pi r_1 \sqrt{h_1^2 + r_1^2} \left(1 - \left(\frac{h_2}{h_1}\right)^2 \right) =$

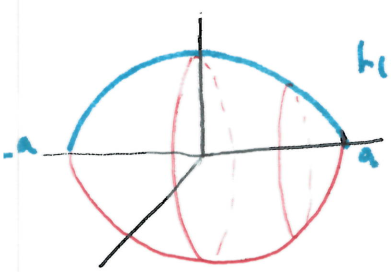
$$= \pi r_1 (L_2 + L) \left(1 - \left(\frac{r_2}{r_1}\right)^2 \right) = \pi r_1 (L_2 + L) \left(\frac{r_1^2 - r_2^2}{r_1^2} \right)$$

$$= \pi (L_2 + L) \left(\frac{r_1 - r_2}{r_1} \right) (r_1 + r_2)$$

$$= \pi (L_2 + L) \left(1 - \frac{r_2}{r_1} \right) (r_1 + r_2)$$

$$= \pi (L_2 + L) \left(1 - \frac{L_2}{L_2 + L} \right) (r_1 + r_2) = \pi L (r_1 + r_2)$$ as we wanted!

Example 2: Sphere of radius a:



$f(x) = \sqrt{a^2 - x^2}$

$f' = \frac{-x}{\sqrt{a^2 - x^2}}$

$$\text{Area}(S) = \int_{-a}^{a} 2\pi \sqrt{a^2 - x^2} \sqrt{1 + \frac{x^2}{a^2 - x^2}}\, dx$$

$$= \int_{-a}^{a} 2\pi a \, dx = 2\pi a (2a) = \boxed{4\pi a^2}$$

Alternative proof (Archimedes). Triangulate the surface & use pyramids with vertex the center of the sphere

Vol pyramid $= \frac{1}{3} a A_k$

$$\boxed{\tfrac{4}{3}\pi a^3} = \text{Vol(Sphere)} \xleftarrow[k\to\infty]{} \sum_k \text{vol pyramids} = \frac{1}{3} a \boxed{\sum_k A_k} \xrightarrow[k\to\infty]{} \boxed{\frac{1}{3} a \,\text{Area}(S)}$$

Compare 2 ends to conclude $\text{Area}(S) = 4\pi a^2$ ✓