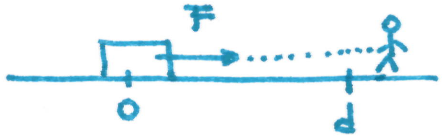
Lecture XXIX: §7.7 Work & Energy

§1 Work:

WORK = "Effort done by a force to move an object (action is don the direction of the movement".

Ex: (1)

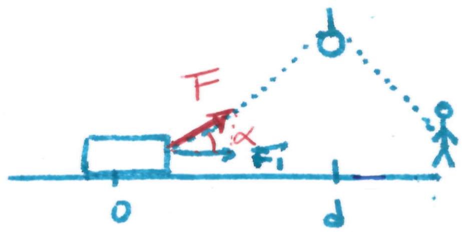


Pull an object with a chord.

F = force, d = distance

$$\boxed{W = F\,d}$$

(2)



Pull an object through a pulley.

. Component of F in the direction of the movement is $F_1 = F\cos\alpha$

so $\boxed{W = (F\cos\alpha)\, d}$

Units: $F = m \cdot a \rightsquigarrow kg \cdot \frac{m}{s^2}$, $lb \cdot \frac{in}{s^2}$ $= (\text{mass}) \frac{(\text{distance})}{(\text{time})^2}$

$W = F \cdot d \rightsquigarrow kg\,\frac{m^2}{s^2}$, $lb\,\frac{in^2}{s^2}$

Eg: Weight = measure of force with which an object is attracted to earth

W = ft-pounds = $\boxed{\text{ft-lbs}}$

. If we lift a box weighing 20 lbs, 3 ft high, the work done is 60-ft-lbs

Other units: (1) cgs (cm-grams-second) $\rightsquigarrow$ F in dynes $= 1\,\frac{cm\,gr}{s^2}$.

(2) mks (m-kg-——) $\rightsquigarrow$ F in Newtons (N) $= 1\,\frac{m\,kg}{s^2}$.

$\Rightarrow$ W in cgs = dyne-cm =: erg

W in mks = N - m =: Joule (J)

Conversions: 1 ft-lb = 1.356 J , 1 J = 10^7 ergs.

Examples above: the Force is the same no matter the location of the object.

Q: What if the force is non-constant? $\rightsquigarrow F = F(x)$ changes with x = distance.

Over a small distance Δx, we can "pretend" F is constant (continuity). So

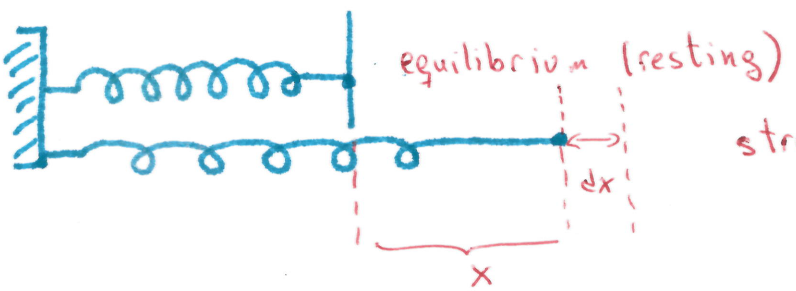
$$\Delta W = F\,\Delta x$$

LINEAR APPROX. $\rightarrow$ $dW \approx F\,dx$ ⟿ FTC

$$\boxed{W_{(u)} = \int_0^u F_{(x)}\,dx \quad \text{for } u \text{ in } \mathbb{R}}$$

§2 Prototypical example 1: SPRINGS

- As we stretch a spring, force pushes it back to equilibrium
- More force acts the further we stretch the spring



F restorative force (against the stretching direction!)

<u>Hooke's Law</u>: $F_{(x)} = k\,x$ where k = spring constant (depends on material & properties of the spring)

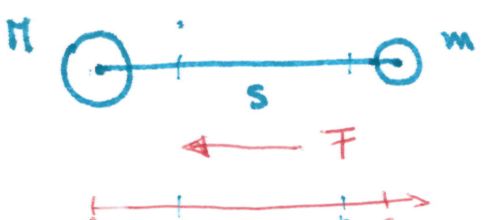
<u>Example</u>: Assume it takes 8 lb of force to hold a 16 in long spring, 2 in away from equilibrium. How much work is done when moving the spring to a 24 in length?

<u>Soln</u> • Use data to find k: $F_{(2)} = k \cdot 2\,in = 8\,lb \Rightarrow \boxed{k = 4\,\frac{lb}{in}}$

• x = distance $= 24 - 16 = 8$

$$W = \left(\int_0^8 4x\,dx\right)\frac{lb}{in} = 2x^2\Big|_0^8\,\frac{lb}{in} = 128\,in^2\,\frac{lb}{in} = \boxed{128\ in\cdot lb}$$

§3 Prototypical example 2: GRAVITY



[Replace each planet with a heavy center]

Force of attraction of 2 bodies of mass M & m with $M > m$. (Larger body is attractor)

$$F_{(s)} = -\frac{G\,M\,m}{s^2}$$

$G > 0$ universal gravity const.

opposite from distance measurement (from larger to lighter)

So $dW = F_{(s)}\,\Delta s = -\dfrac{GMm}{s^2}\,ds$

Moving m from $s=a$ to $s=b$ against the force of gravity requires:

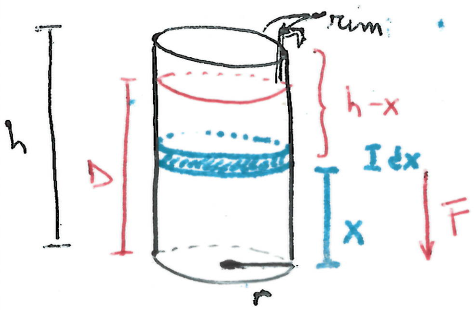
$$W = -\int_a^b F(s)\,ds = \int_a^b \frac{GMm}{s^2}\,ds = GMm\left(\frac{-1}{s}\Big|_a^b\right)$$

$$= -GMm\left(\frac{1}{b}-\frac{1}{a}\right) = \begin{cases} >0 & \text{if } b>a \\ <0 & \text{if } b<a \end{cases}$$

Remark: If $b \to \infty$ then $W \xrightarrow[b\to\infty]{} \frac{GMm}{a} > 0$

NOTE: potential of 2 objects (work required to separate the 2 objects completely)

§4 Prototypical example 3: PUMPING WATER from a tank



- Cylindrical tank of radius $= r$ & height $= h$
- Filled with water to depth D
- Body to "move" = drops of water.
- w = weight-density of water = weight/unit vol = lb/in^3

Q: How much work is done to pump the water ~~OUT of~~ over the rim of the tank?

- Key assumption: Work is the same for all drops of water at the same distance below the rim (so it only depends on $h-x$).

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Vol $= \pi r^2 dx$ so weight of this is computed as

$$F(x) = \text{density}\cdot\text{Vol} = w\cdot\pi r^2 dx$$

We travel distance $h-x$ to pump water out:

$$dW = F(x)(h-x) = w\pi r^2(h-x)\,dx$$

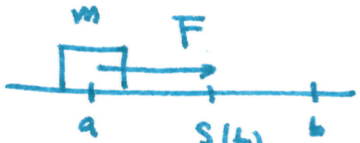
So $$W = \int_0^D w\pi r^2(h-x)\,dx = w\pi r^2\left(hx-\frac{x^2}{2}\right)\Big|_0^D = \boxed{w\pi r^2 D\left(h-\frac{D}{2}\right)}$$

- Measuring from the top gives the same result: [$\tilde{x} = h-x$, $d\tilde{x} = -dx$; flipped limits of integration]

$$W = \int_{h-D}^{h} w\pi r^2 x\,dx = w\pi r^2\frac{x^2}{2}\Big|_{h-D}^{h} = \frac{w\pi r^2}{2}\left(h^2-(h-D)^2\right)$$
$$= w\pi r^2 D\frac{(2h-D)}{2} \checkmark$$

§5 Energy

• KINETIC ENERGY: energy due to motion $= \frac{1}{2} m v^2$



$F = m a_{(t)} = m \frac{dv}{dt}$ and $v = \frac{ds}{dt}$ velocity

(F moves the object along a straight line!)

Theorem: The work done by F equals the change in the kinetic energy of the particle

Proof Use the Chain Rule to write $F = m \frac{dv}{dt} = m \frac{dv}{ds} \frac{ds}{dt} = m v \frac{dv}{ds}$

(Like we did when discussing escape velocity).

$$W = \int_a^b F(s)\, ds = \int_a^b m v \frac{dv}{ds}\, ds \underset{\substack{\downarrow \\ \text{change from} \\ s \text{ to } v}}{=} \int_{v_a}^{v_b} m v\, dv = \frac{1}{2} m v^2 \Big|_{v_a}^{v_b}$$

$$= \frac{1}{2} m \left(v_b^2 - v_a^2\right) \quad \square$$

• POTENTIAL ENERGY $= V(s) = -\int F(s)\, ds$ (if F is cont.) $\frac{dV}{ds} = -F(s)$

Then $W_{(a \to s)} = -\left(V(s) - V(a)\right) = V(a) - V(s)$

(work done by force when moving from a to s

|| → Thm

$\frac{1}{2} m \left(v_s^2 - v_a^2\right)$

So $V(a) - V(s) = \frac{1}{2} m v_s^2 - \frac{1}{2} v_a^2$ & regroup (a vs s)

$$\text{Energy} := \underbrace{V(a) + \frac{1}{2} v_a^2}_{\text{independent of } s_{(t)}!} = V(s) + \frac{1}{2} m v_s^2$$

↑ TOTAL ↑ potential energy ↑ kinetic energy

$\boxed{\text{LAW OF CONSERVATION OF ENERGY}}$

TOTAL energy of the particle is conserved at every stage of the movement.

Nice application (Working heart): work of a single stroke is $W = 0.74$ ft-lb.

= pump.