Lecture XXX §8.2 Exponents & Logarithms
§8.3 The Number e and the function $y = e^x$

§1 Basic on exponents

$a > 0$, x in $\mathbb{R}$ ⇝ Q: What is a^x?

Special cases ① x integer

1) $x = n > 0$ $\quad a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$

2) $x = 0$ $\quad a^0 = 1$

3) $x = -n < 0$ $\quad a^{-n} = (a^{-1})^n = \underbrace{\frac{1}{a} \cdots \frac{1}{a}}_{n \text{ times}}$

② x rational, write $x = \frac{m}{n}$ fraction in lowest terms, with $n > 0$. Then

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ where $\sqrt[n]{a}$ is the unique positive number b with $b^n = a$.

General definition: Approximate x via rational numbers arbitrarily close to them and define via a limiting process

Def $\boxed{a^x = \lim_{\substack{r \to x \\ r \text{ rat'l}}} a^r.}$ for $a > 0$, any real x.

GOOD NEWS: By construction the function will be automatically continuous.

BAD NEWS: It's not clear that the function is well-defined, meaning it is independent of the choice of $r \to x$.

(Eg: if x is rational we can take $r = x$, but we may take other rationals. How do we know this other choice gives us our original definition?)

All works out in the end (& the reason is mainly the law of exponents)

Law of exponents:

(1) $a^x a^y = a^{x+y}$, (2) $a^{x-y} = \frac{a^x}{a^y}$ (3) $(a^x)^y = a^{xy}$

Proof: By limit laws, & definition of a limit we may assume x, y are rationals (since $r \to x$, $r' \to y$ then $\underbrace{r + r'}_{\text{in } \mathbb{Q}} \to x + y$, $\underbrace{r - r'}_{\text{in } \mathbb{Q}} \to x - y$, $\underbrace{r \cdot r'}_{\text{in } \mathbb{Q}} \to xy$)

(1) – (3) are true for x, y integers by definition.

Claim: $a^{\frac{m}{n}} = a^{\frac{km}{kn}}$ (in short: we don't need the lowest terms in)

Pf/ $a^{\frac{km}{kn}} = \left(\sqrt[kn]{a}\right)^{km} \underset{\downarrow}{=} \left(\left(\left(\left(\sqrt[n]{a}\right)^{1/k}\right)^{k}\right)^{m} = \left(\sqrt[n]{a}\right)^{m} = a^{\frac{m}{n}}$.

- By uniqueness $\sqrt[k]{\sqrt[n]{a}} = \sqrt[kn]{a}$
- (3) for k, m integers

. To show (1) — (3) for rational exponents, we choose to write $x = \frac{m}{n}$, $y = \frac{l}{n}$ (take common denominator between x & y).

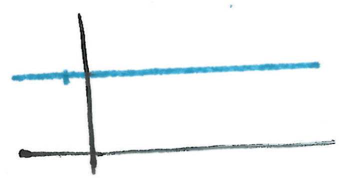
(1) $a^{\frac{m}{n}} a^{\frac{l}{n}} = \left(\sqrt[n]{a}\right)^m \left(\sqrt[n]{a}\right)^l \underset{\substack{\downarrow \\ b = \sqrt[n]{a} > 0}}{=} b^m b^l \underset{\substack{\downarrow \\ \text{(1) for integer exponents}}}{=} b^{m+l} = a^{\frac{m+l}{n}}$.

(2) $a^{\frac{m}{n}} / a^{\frac{l}{n}} = \dfrac{\left(\sqrt[n]{a}\right)^m}{\left(\sqrt[n]{a}\right)^l} \underset{\substack{\downarrow \\ b = \sqrt[n]{a} > 0}}{=} \dfrac{b^m}{b^l} \underset{\substack{\downarrow \\ \text{(2) for integer exp}}}{=} b^{m-l} = a^{\frac{m-l}{n}}$.

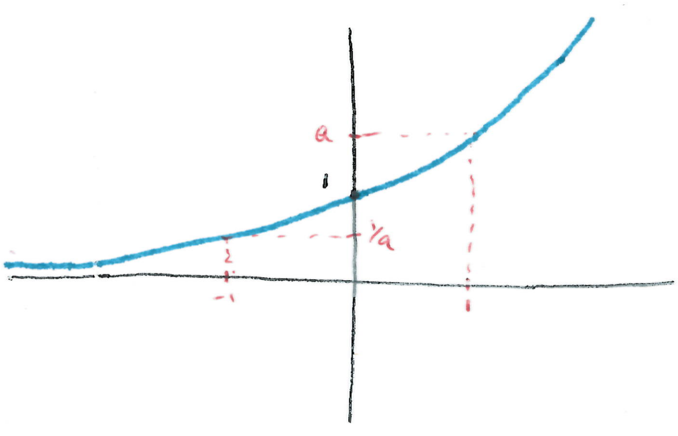
(3) $\left(a^{\frac{m}{n}}\right)^{\frac{l}{n}} = \left(\sqrt[n]{\left(\sqrt[n]{a}\right)^m}\right)^l \underset{\substack{\downarrow \\ \text{uniqueness}}}{=} \left(\left(\sqrt[n]{\sqrt[n]{a}}\right)^m\right)^l \underset{\substack{\downarrow \\ \text{uniqueness}}}{=} \left(\left(\sqrt[n^2]{a}\right)^m\right)^l \underset{\substack{\downarrow \\ b = \sqrt[n^2]{a} \\ \text{(3) for int exp.}}}{=} \left(\sqrt[n^2]{a}\right)^{m \cdot l} = a^{\frac{ml}{n^2}}$.

Graphs of a^x:

- If $a = 1$: $a^x = 1$ for all x



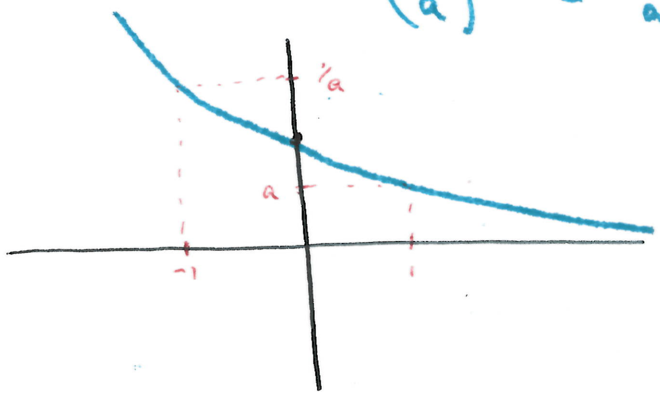
- If $a > 1$: . a^x is continuous, $a^r > 1$ for any r positive rational, so $a^x > 1$ for any $x > 0$ real
  - a^x is st increasing (if $x < y$ $a^y = a^{x + (y-x)} = a^x a^{\overset{>0}{\overbrace{y-x}}} > a^x$)
  - $\lim\limits_{x \to \infty} a^x = \infty$, $\lim\limits_{x \to -\infty} a^x = \lim\limits_{x \to \infty} \frac{1}{a^x} = 0$.



. Range $= \{ y > 0 \}$

. str. increasing $\leadsto$ injective! (horiz line Test)

• If $a<1$ $\quad a^x = \left(\frac{1}{a}\right)^{-x}$ & $\frac{1}{a}>1$ $\rightsquigarrow$ properties & graphs get affected



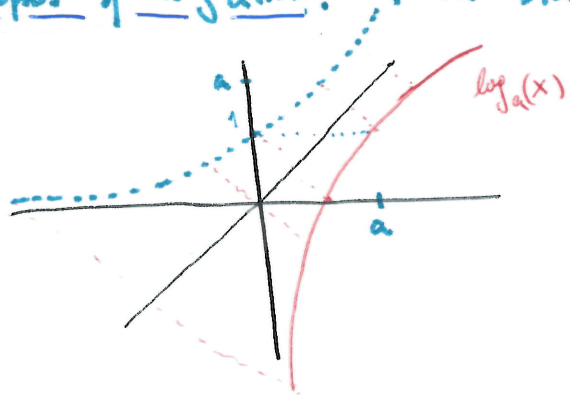
- a^x is cont, sti. decreasing
- $\lim_{x\to\infty} a^x = 0$; $\lim_{x\to-\infty} a^x = \infty$
- Range $= \{y > 0\}$
- injective!

Conclusion: For $a \neq 1$, $a^x : \mathbb{R} \to \{y>0\}$ is injective & surjective

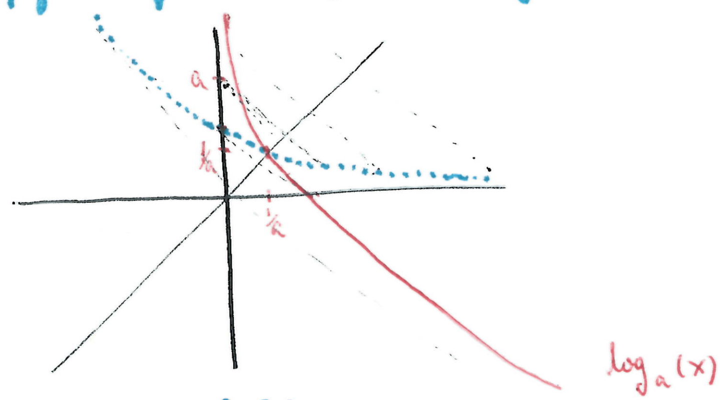
We can find an inverse function $= \log_a : \{x>0\} \to \mathbb{R}$
$\qquad x \longmapsto \log_a(x)$

Call it logarithm to the base a

Graphs of $\log_a(x)$: Turn over the figure for a^x about $x=y$ line



$a>1$



$a<1$

$\left[\log_{1/a}(x) = -\log_a(x)\right]$ $\quad a^y = x \Rightarrow x = a^y = \left(\frac{1}{a}\right)^{-y}$

Basic properties of $\log_a$:

(1) $\log_a(x_1 x_2) = \log_a x_1 + \log_a x_2$

($y_1 = \log_a x_1$, $y_2 = \log_a x_2$ $\rightsquigarrow$ $x_1 = a^{y_1}$, $x_2 = a^{y_2}$
so $x_1 x_2 = a^{y_1} a^{y_2} = a^{y_1+y_2}$)

(2) $\log_a(x_1/x_2) = \log_a x_1 - \log_a x_2$

($x_1 = a^{y_1}$, $x_2 = a^{y_2}$ ~~so~~ $\frac{x_1}{x_2} = a^{y_1}/a^{y_2} = a^{y_1 - y_2}$)

(3) $\log_a(x^b) = b \log_a(x)$

($a^y = x$ so $x^b = (a^y)^b = a^{yb}$ so $\log_a x^b = by$)

(4) $\log_a(a^x) = x$ by def ($y = \log_a a^x$ if $a^y = a^x$ But this forces $y = x$)

(5) $a^{\log_a x} = x$ by def ($y = \log_a x$ means $a^y = x$)

(6) $\log_a a = 1$, $\log_a 1 = 0$ $(a^0 = 1)$.

Note: If $a > 1$: $\lim\limits_{x\to\infty} \log_a x = \infty$, $\lim\limits_{x \to 0} \log_a x = -\infty$

If $a < 1$: $\lim\limits_{x\to\infty} \log_a x = -\infty$, $\lim\limits_{x\to 0} \log_a x = +\infty$.

§2 Derivatives of exponentials

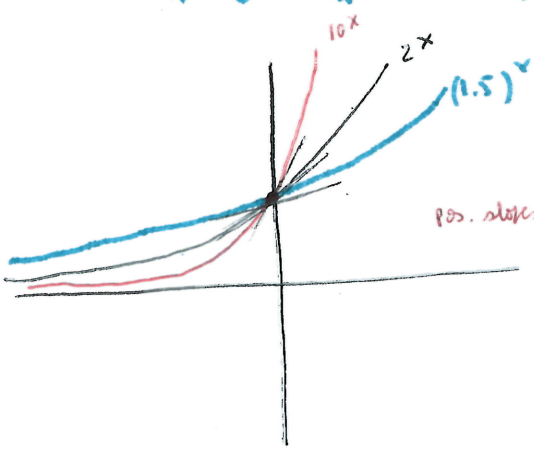
We compute $\frac{d}{dx} a^x$ for $a > 0$ using the method of increments

$$\frac{d}{dx} a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} a^x \left(\frac{a^{\Delta x} - 1}{\Delta x} \right) = a^x \boxed{\lim_{h\to 0} \frac{a^h - 1}{h}}$$

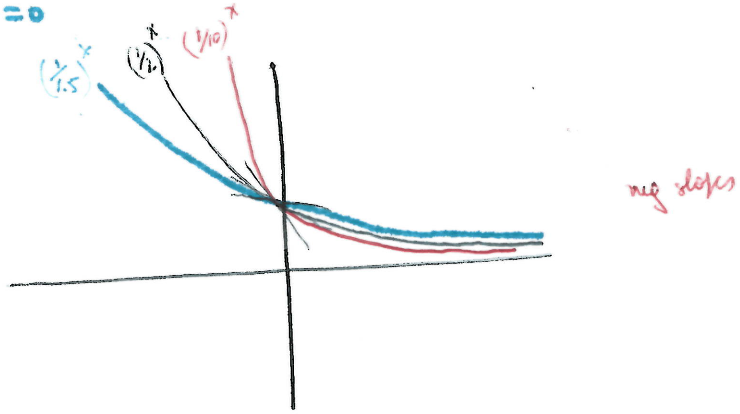
constant depending on a

as long as $\lim\limits_{h\to 0} \frac{a^h - 1}{h}$ exists!

But $\lim\limits_{h\to 0} \frac{a^h - 1}{h} = \frac{d}{dx} a^x \Big|_{x=0}$ = slope of the tangent line to a^x at $(0,1)$



$a > 1$.



$a < 0$

Def: e is the unique real number satisfying $\lim\limits_{h\to 0} \frac{e^h - 1}{h} = 1$ (Tang has slope 1!)

Consequence $\frac{d}{dx} e^x = e^x$. Call e^x the exponential function

($e \sim 2.7182\ldots$ more in Appendix A8)

$e \sim (1+h)^{1/h}$ as $h \to 0$.

Note $\frac{d}{dx}(ce^x) = ce^x$ for any constant c.

Prop: All solutions to $y' = y$ are of the form $y = ce^x$ for some parameter c

Proof: given a solution $f(x)$, consider $g(x) = \frac{f(x)}{e^x}$. By Quotient Rule

$$\frac{d}{dx}(g(x)) = \frac{f' e^x - f e^x}{(e^x)^2} = \frac{f e^x - f e^x}{e^{2x}} = 0 \quad \text{so } g(x) \text{ is constant} = c!$$

Prop: $(e^x)' = e^x$ so $\int e^x \, dx = e^x + C$ $\rightsquigarrow$ one more building block!

Eg: $\int e^{5x} dx \underset{\substack{u=5x \\ du = 5dx}}{=} \int e^u \frac{du}{5} = \frac{1}{5}(e^u + c) = \frac{e^{5x}}{5} + \tilde{C}$.

$\int x e^{x^2} dx \underset{u = x^2}{=} \int e^u \frac{du}{2} = \frac{e^{x^2}}{2} + C$.

§3 The natural logarithm:

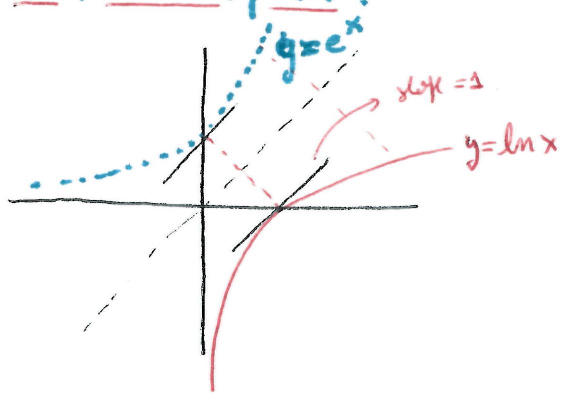
Def. $\ln(x) = \log_e(x)$ so $y = \ln x$ means $e^y = x$

• **Prop**: $\ln(x)$ is infinitely differentiable and $\frac{d}{dx} \ln x = \frac{1}{x}$ $\rightsquigarrow$ $\boxed{d \ln x = \frac{dx}{x}}$

Why? Use implicit differentiation on $e^y = x$ $\rightsquigarrow$ $e^y y' = 1$ so $y' = \frac{1}{e^y} =$

$y' = \frac{1}{x}$ □

• **Properties of ln**:



• Slopes of tangent lines for e^x & $\ln x$ are related!

• $\lim_{x \to 0^+} \ln x = -\infty$ $\quad$ $\lim_{x \to \infty} \ln x = \infty$

(because $e > 1$)

Integration: $\int \frac{dx}{x} = \ln|x| + C$ $\rightsquigarrow$ one more building block.

Examples: $\int \frac{x^3}{x^4+1} dx \underset{u = x^4+1}{=} \frac{1}{4} \int \frac{du}{u} = \frac{1}{4} \ln(x^4+1) + C$

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx \underset{u = \cos x}{=} \int \frac{-du}{u} = -\ln(u) + C = -\ln(\cos x) + C$

as $\cos x > 0$