Lecture XXXI : §8.4 The natural logarithm function
§8.5 Population growth & radioactive decay

§1 Other bases of exponentials & logarithms

- $a^x = (e^{\ln a})^x = e^{x \ln a}$ so $\frac{d}{dx} a^x = e^{x\ln a} \ln a = \boxed{a^x \ln a}$

Last time: $\frac{d}{dx} a^x = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$ so $\boxed{\ln a = \lim_{h\to 0} \frac{a^h-1}{h}}$

- Similarly: $y = \log_a x$ means $x = a^y = (e^{\ln a})^y = e^{y \ln a}$

So $\ln x = y \ln a \ln e = y \ln a$ gives $\boxed{\log_a x = \frac{\ln x}{\ln a}}$

In particular: $\frac{d}{dx} \log_a x = \frac{1}{\ln a} \frac{1}{x}$

Q: Growth of e^x & $\ln x$?

A
- $\lim_{x\to\infty} \frac{e^x}{x^n} = +\infty$ for any integer (so e^x grows faster than ANY polynomial)
- $\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$ for all $p > 0$ (so $\ln x$ grows slower than ANY non constant polynomial)

§2 Solving Differential Equations

Last Time: All solutions to $y' = y$ are of the form $y_{(x)} = c e^x$ for some c.

Prop: $y' = k y$ has solutions $y = c e^{kx}$ for c constant
(again, we have a 1-parameter family of solutions)

Proof Separation of variables $\frac{y'}{y} = k \rightsquigarrow \int \frac{dy}{y} = \int k \, dx$

gives $\ln y = kx + C'$

Take exponential so $y = e^{\ln y} = e^{kx + C'} = e^{C'} e^{kx} = \boxed{\overset{>0}{\tilde{C}}} e^{kx}$

Other solutions: $f(x) = \frac{y}{e^{kx}}$ $f'(x) = \frac{y' e^{kx} - y k e^{kx}}{e^{2kx}} = \frac{k y e^{kx} - k y e^{kx}}{e^{2kx}} = 0$

So $y(x) = C e^{kx}$ for some C constant (of any sign!)

§ 2 Population Growth

- **Basic model**: $N(t)$ = population at time t (e.g. bacteria)

  Assumptions: unlimited food, no predators, no deaths (toy model)

  Rate of change of current population: $\boxed{\dfrac{dN}{dt} = k\,N(t)}$ for some constant k.

  k = % population increase

  Soln: $\boxed{N_{(t)} = C e^{kt}}$ where $\boxed{C = N_0 = \text{population at time } t=0}$

- **k vs doubling time**

  t_d = doubling time = time it takes for the population to double in size.

  Assume $N_0 > 0$. So $2N_0 = N(t_d) = N_0 e^{k t_d}$ so $2 = e^{kt_d}$

  Get $\boxed{kt_d = \ln 2}$ $\rightsquigarrow$ $t_d = \dfrac{\ln 2}{k}$ or $k = \dfrac{\ln 2}{t_d}$.

Remark: $N(t+t_d) = N_0 e^{k(t+t_d)} = \underbrace{N_0 e^{kt}}_{=N(t)} \overbrace{\boxed{e^{kt_d}}}^{=2} = N(t)\,2$

So it doesn't matter when we start counting. The doubling time is the same!

Example 1: The number of bacteria in a culture doubles ~~every~~ hour. How long does it take for 1000 bacteria to produce 1 billion $= 10^9$?

Soln: $t_d = 1$ hour so $k = \dfrac{\ln 2}{t_d} = \ln 2$ $\rightsquigarrow$ $N(t) = N_0\, e^{kt}$

Take $N_0 = 1000$ so $10^9 = N(t) = 10^3 e^{(\ln 2)t} = 10^3 e^{t \ln 2}$ $\rightsquigarrow$ $10^6 = e^{t\ln 2}$

Take $\ln$: $6 \ln 10 = t \ln 2$ $\rightsquigarrow$ $t = \dfrac{6\ln 10}{\ln 2}$

Example 2: In 1970, the world population was 3.6 billion. The Earth weighs $6586 \cdot 10^{18}$ Tn. If the population increases at a rate of 2% per year & an average person weighs 120 lbs, when will the weight of all people equal the Earth's weight?

Soln: $k = \dfrac{2}{100}$ & $N_0 = 3.6 \cdot 10^9$ so $N(t) = 3.6\; 10^9\, e^{\frac{2}{100}t}$

Want $120\, N(t) \stackrel{?}{=} 6586\; 10^{18}\; 2000$ (1 Tn = 2000 lb)

$120\;\; 3.6\, 10^9\, e^{2t/100} = 6586\; 10^{18}\, 2000 = 6586\; 10^{21} \cdot 2$

So $e^{t/50} = \dfrac{6586 \cdot 2}{12 \quad 36}\, 10^{21-9} = \dfrac{3293}{108}\, 10^{12}$ $\rightsquigarrow$ $t = 50 \ln\left(\dfrac{3293}{108}\, 10^{12}\right) =$

Conclusion: $\boxed{1552.42 \text{ years}}$ $\boxed{1552.42} = 50(\ln 3293 + 12 \ln 10 - \ln 108)$

§3 Radioactive decay

Characteristic feature of radioactive materials: instead of growth, we have decay

We write $\boxed{\dfrac{dx}{dt} = -kx}$ for some $k>0$ (decay constant)

Solution: $\boxed{x(t) = C\,e^{-kt}}$ where $\boxed{C = x_0}$ = amount of material at time $t=0$

Note: $x(t) \neq 0$ for all t, so radioactive materials NEVER completely decay.

Analog of doubling time is the half-time $= t_{1/2}$ = time it takes for the substance to decay to half its original amount.

$$x(t_{1/2}) = \tfrac{1}{2}x(0) = \tfrac{1}{2}x_0 \rightsquigarrow e^{-kt_{1/2}} = \tfrac{1}{2} = 2^{-1}$$

SAME as with doubling time

$$-kt_{\frac{1}{2}} = -\ln 2 \rightsquigarrow \boxed{kt_{\frac{1}{2}} = \ln 2}$$

Remark: $x(t+t_{1/2}) = x_0\,e^{-k(t+t_{1/2})} = \underbrace{x_0 e^{-kt}}_{=x(t)}\,\underbrace{e^{-kt_{1/2}}}_{=\frac{1}{2}} = \tfrac{1}{2}x(t)$

so the initial time is irrelevant.

Example 3: Cesium 137 decays to 20% in 10 years. What's its half-time?

Soln: $x_0 e^{-k10} = x(10) = \dfrac{20}{100}x_0 = \dfrac{x_0}{5}$ so $e^{-10k} = 5^{-1}$

$$-10k = -\ln 5 \rightsquigarrow \boxed{k = \frac{\ln 5}{10}}$$

So $t_{\frac{1}{2}} = \dfrac{\ln 2}{k} = \dfrac{\ln 2}{\ln 5/10} = 10\dfrac{\ln 2}{\ln 5} \simeq 4.3067$ years

Main application: Radiocarbon dating (Libby's 1940s)

- Carbon 14 in a living thing starts decaying right after its death.
- —— has a half-time of ~5600 years

Eg: If a piece of old wood has $\frac{1}{2}$ radioactivity from Carbon 14 as a living tree has, then it lived about 5,600 years ago. If it has $\frac{1}{4}$ of radioactivity, then it lived 11,200 years ago, etc.

[This has been verified in Sequoia trees, furniture from Egyptian tombs whose age is known by other means.]