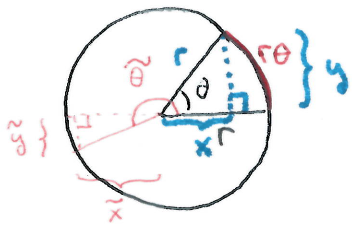
Lecture XXXII : §9.1 Review of Trigonometry
§9.2 The derivative of sin & cosine
§9.3 Integrals of sin x & cos x
§9.4 The other 4 trigonometric functions

§1 Basics:



- θ measured in deg between 0 & 360°
- ________ radians between 0 & 2π.

(1 radian = angle to describe an arc in unit circle of length 1)

- If radius is r & angle is θ (in rad), arc has length $r\theta$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{for any } 0 \le \theta \le 2\pi$$

$$\begin{cases} \tilde{x} = r\cos\tilde{\theta} < 0 & \text{because } \pi < \tilde{\theta} < \frac{3}{2}\pi \\ \tilde{y} = r\sin\tilde{\theta} < 0 & \text{\_\_\_\_\_\_} \end{cases}$$

Periodicity : $\cos(\theta + 2\pi) = \cos\theta$
$\sin(\theta + 2\pi) = \sin\theta$

4 more functions : $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot(\theta) = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$,
$\csc\theta = \dfrac{1}{\sin\theta}$.

Parity : $\left.\begin{array}{l}\sin(-\theta) = -\sin\theta \\ \tan(-\theta) = -\tan\theta\end{array}\right\}$ (ODD)
$\cos(-\theta) = \cos\theta$ (EVEN)

Area (sector) : $\pi r^2 \dfrac{\theta}{2\pi} = \dfrac{\theta r^2}{2}$

Length (arc) : $\theta r = 2\pi r \dfrac{\theta}{2\pi}$.

§2 Trig Identities

① $\boxed{\sin^2\theta + \cos^2\theta = 1}$

$$\begin{cases} \tan^2\theta + 1 = \sec^2\theta & (1/\cos^2\theta) \\ 1 + \cot^2\theta = \csc^2\theta & (1/\sin^2\theta) \end{cases}$$

② Addition formulas

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$$

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

divide by $\cos\theta\cos\varphi$ num & denom

$$\text{So } \tan(\theta + \varphi) = \frac{\sin\theta\cos\varphi + \cos\theta\sin\varphi}{\cos\theta\cos\varphi - \sin\theta\sin\varphi} \stackrel{\uparrow}{=} \frac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi}$$

Alternative (using complex numbers)

Define i as $\sqrt{-1}$ so $\boxed{i^2=-1}$

Define $e^{i\theta} := \cos\theta + i \operatorname{sen}\theta$

so $e^{i(\theta+\varphi)} = \cos(\theta+\varphi)1 + i \operatorname{sen}(\theta+\varphi)$

$\parallel \to$ exp Laws

$$e^{i\theta}e^{i\varphi} = (\cos\theta + i\operatorname{sen}\theta)(\cos\varphi + i\operatorname{sen}\varphi)$$

$$\underset{\text{Distribute}}{=} \cos\theta\cos\varphi + i\cos\theta\operatorname{sen}\varphi + i\operatorname{sen}\theta\cos\varphi + \overset{-1}{i^2}\operatorname{sen}\theta\operatorname{sen}\varphi$$

$$\underset{\text{regroup}}{=} (\cos\theta\cos\varphi - \operatorname{sen}\theta\operatorname{sen}\varphi)1 + i(\cos\theta\operatorname{sen}\varphi + \operatorname{sen}\theta\cos\varphi)$$

so coeff of i & 1 on top & bottom must agree. We get

$$\begin{cases}\cos(\theta+\varphi) = \cos\theta\cos\varphi - \operatorname{sen}\theta\operatorname{sen}\varphi\\ \operatorname{sen}(\theta+\varphi) = \cos\theta\operatorname{sen}\varphi + \operatorname{sen}\theta\cos\varphi.\end{cases}$$

Special cases:

① Double angle: (*)
$$\boxed{\begin{aligned}\operatorname{sen}2\theta &= 2\operatorname{sen}\theta\cos\theta\\ \cos 2\theta &= \cos^2\theta - \operatorname{sen}^2\theta\end{aligned}} = \cos^2\theta - (1-\cos^2\theta) = \boxed{2\cos^2\theta - 1}$$

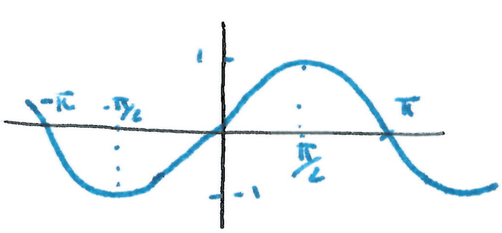
Also $\cos 2\theta = 2\cos^2\theta - 1 = 2(1-\operatorname{sen}^2\theta) - 1 = \boxed{1 - 2\operatorname{sen}^2\theta}$

(*) most useful for integration:

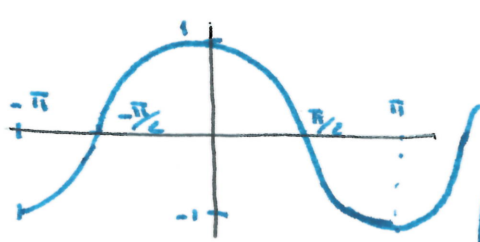
② Half-angle:
$$\boxed{\begin{aligned}2\cos^2\theta &= 1 + \cos 2\theta\\ 2\operatorname{sen}^2\theta &= 1 - \cos 2\theta\end{aligned}}$$

means $\boxed{\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \quad \& \quad \operatorname{sen}^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}}$

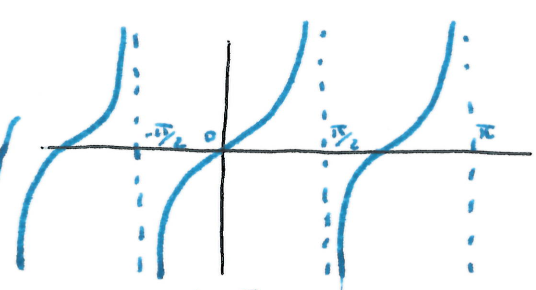
§ 3 Graphs



$y = \operatorname{sen}\theta$

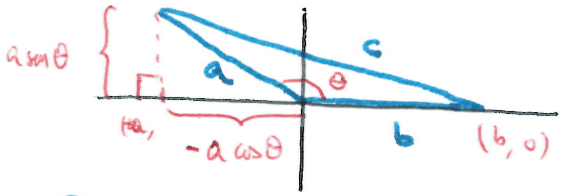


$y = \cos\theta$



$y = \tan\theta$
[Periodicity $= \pi$]

§4 Law of cosines



Law of cosines: $\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}$

$\frac{\pi}{2} \leq \theta \leq \pi$

Proof:
$$
\begin{aligned}
c^2 &= (a\,\text{sen}\,\theta)^2 + (b - a\cos\theta)^2 \\
&= a^2\,\text{sen}^2\theta + b^2 + a^2\cos^2\theta - 2ab\cos\theta \\
&= a^2\underbrace{(\text{sen}^2\theta + \cos^2\theta)}_{=1} + b^2 - 2ab\cos\theta \\
&= a^2 + b^2 - 2ab\cos\theta \quad \checkmark
\end{aligned}
$$

Note: For $\theta = \frac{\pi}{2}$ we recover Pythagoras' Thm since $\cos\frac{\pi}{2} = 0$.

§5 Derivatives & Integrals

- $\frac{d}{d\theta}\cos\theta = -\text{sen}\,\theta$
- $\frac{d}{d\theta}\text{sen}\,\theta = \cos\theta$

$\frac{d}{d\theta}\tan\theta = \sec^2\theta$

$\frac{d}{d\theta}\cot(\theta) = -\csc^2\theta$

$\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$

$\frac{d}{d\theta}\csc\theta = -\csc(\theta)\cot\theta$ $\left(= -\frac{\cos\theta}{\text{sen}^2\theta}\right)$

Proof: Quotient rule.

- $\int \cos\theta\, d\theta = \text{sen}\,\theta + C$
- $\int \text{sen}\,\theta\, d\theta = -\cos\theta + C$
- $\int \sec\theta\tan\theta\, d\theta = \sec\theta + C$

$\int \sec^2\theta\, d\theta = \tan\theta + C$

$\int \csc^2\theta\, d\theta = -\cot\theta + C$

$\int \csc(\theta)\cot\theta\, d\theta = -\csc\theta + C$

Extra formulas:

① $\int \tan x\, dx = \int \frac{\text{sen}\,x}{\cos x}\, dx \overset{u=\cos x}{=} \int -\frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$

② $\int \cot x\, dx = \int \frac{\cos x}{\text{sen}\,x}\, dx \overset{u=\text{sen}\,x}{=} \int \frac{du}{u} = \ln|\text{sen}(x)| + C$

③ $\int \sec x\, dx = \ln(\sec(x) + \tan(x)) + C$

Why? $\int \sec x \, dx = \int \sec x \dfrac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \dfrac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$

$$\underset{u = \sec x + \tan x}{=} \int \frac{du}{u} = \ln u + C = \ln(\sec x + \tan x) + C.$$

$du = (\sec x \tan x + \sec^2 x)\, dx$

④ $\int \csc(x)\, dx = -\ln(\csc(x) + \cot(x)) + C$

Why? $\int \csc x \, dx = \int \csc x \dfrac{\csc(x) + \cot(x)}{\csc(x) + \cot(x)} \, dx = \int \dfrac{\csc^2(x) + \csc(x)\cot(x)}{\csc(x) + \cot x} \, dx$

$$\underset{u = \csc(x) + \cot(x)}{=} - \int \frac{du}{u} = -\ln(\csc(x) + \cot(x)) + C$$

$du = (-\csc(x)\cot(x) - \csc^2(x))\, dx$

⑤ $\int \cos^2 x \, dx = \int \dfrac{1 + \cos 2x}{2} \, dx = \dfrac{1}{2} x + \dfrac{\sin 2x}{4} + C$

$\int \sin^2 x \, dx = \int \dfrac{1 - \cos 2x}{2} \, dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4} + C.$