Lecture XXXIII §9.5 The inverse trigonometric functions

GOAL: Find antiderivatives of the following 2 functions:

(1) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = ??$ & (2) $\displaystyle\int \frac{dx}{1+x^2} = ??$

A: Use "trigonometric substitutions".

(1) Pick: $x = \sin(u)$ so $dx = \cos(u)\,du$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \sqrt{\cos^2 u} = \cos u.$$

So $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos u\,du}{\cos u} = \int 1\,du = u + C = \sin^{-1} x + C$ ⇝ need to find an inverse function for $\sin u$

(2) Pick: $x = \tan u$ so $dx = \sec^2 u\,du$

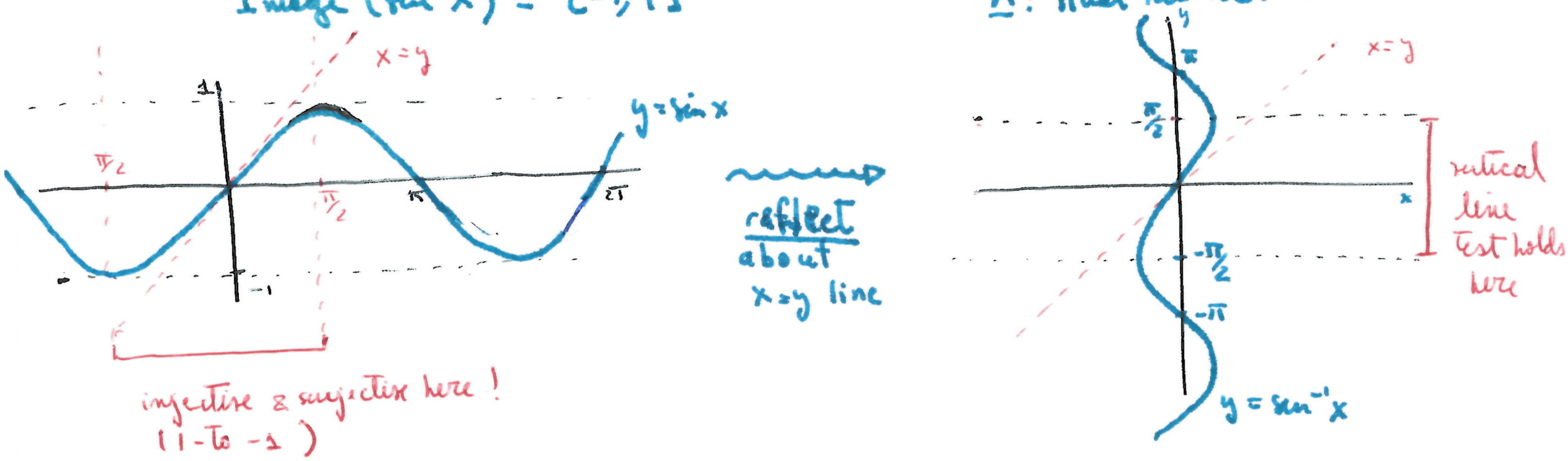
$$1+x^2 = 1+\tan^2 u = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = \frac{1}{\cos^2 u} = \sec^2 u$$

So $\displaystyle\int \frac{dx}{1+x^2} = \int \frac{\sec^2 u\,du}{\sec^2 u} = \int du = u + C = \tan^{-1} x + C$ ⇝ need to invert $\tan u$!

Finding inverses to $\sin x$ & $\tan x$ is forced on us!

§1 Inverse sine $= \sin^{-1} x$ or $\arcsin x$ (2 notations)

Recall: $\sin(x)$ is periodic with period 2π ⇝ Q: How to invert?
Image $(\sin x) = [-1, 1]$ A: Must restrict the domain!

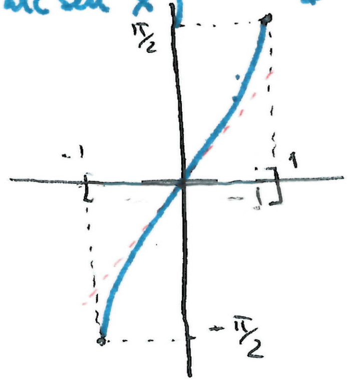


Def: $\arcsin : [-1, 1] \longrightarrow \mathbb{R}$ $\quad \arcsin x = y$ if $\sin y = x$.

$\left.\begin{array}{l} \text{Image}(\arcsin) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \text{Domain}(\arcsin) = \text{Image of } \sin = [-1, 1] \end{array}\right\}$ Write $\arcsin : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Properties $x = \sin(\arcsin x)$ & $x = \arcsin(\sin x)$

Graph of arc sin



Remark: By construction graph of arc sin is smooth (Tan lines come from rotating Tang lines to sin x about $x=y$ line). Furthermore:

Prop: arc sin x is differentiable & $\boxed{\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}}$.

Why? Use implicit differentiation.

$y = \arcsin x$ means $\sin y = x$. $\xrightarrow{\frac{d}{dx}}$ $\cos y \cdot y' = 1$

$$y' = \frac{1}{\cos y}$$

But since y in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos y \geq 0$ so $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

We get. $y' = \frac{1}{\sqrt{1-x^2}}$. ✓

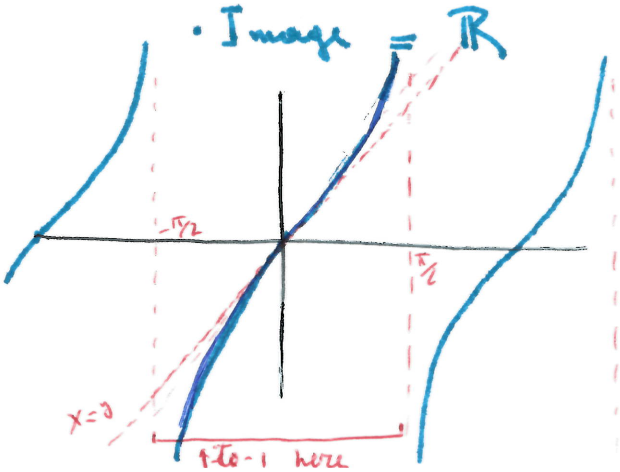
Consequence:
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.}$$

⚠ From $y' = \frac{1}{\sqrt{1-x^2}}$ we see that $|x|<1$ is forced on us. So we don't consider derivatives at $x = \pm 1$.

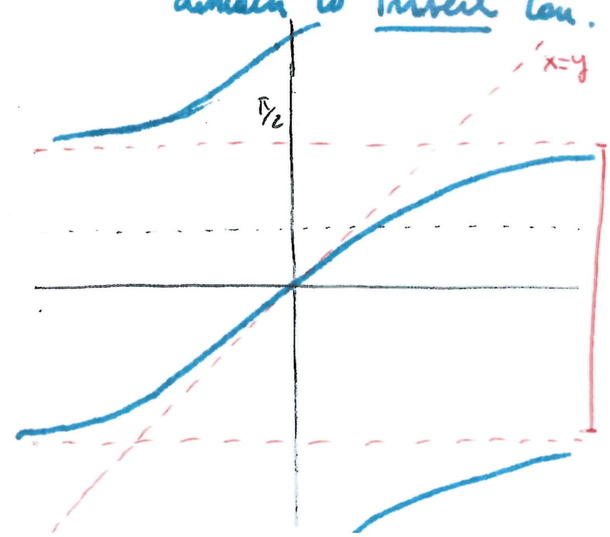
§2 Inverse tangent

$= \arctan x$ or $\tan^{-1} x$ (2 notations)

Recall:
- $\tan x$ is periodic with period π ⇝ Need to restrict the domain to invert tan.
- vertical asymptotes at $\frac{\pi}{2}, \frac{3\pi}{2}, \dots$
  $-\frac{\pi}{2}, -\frac{3\pi}{2}, \dots$
- Image $= \mathbb{R}$



⇝ reflect about $x=y$



<u>Conclusion:</u> We need to pick a branch to define arctan, as we did with sin.

<u>Def</u> $\tan^{-1}(x) = \arctan(x) : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\qquad y = \arctan x$

$x \longmapsto \arctan(x)$ $\qquad$ means $x = \tan y$

<u>Properties:</u> • $\tan x$ is increasing, so is $\arctan(x)$

• $\arctan(0) = 0$

• $x = \arctan(\tan x)$ & $x = \tan(\arctan x)$

• $\tan x$ has well-defined tangent lines, then so does $\arctan(x)$.

<u>Prop:</u> $\arctan x$ is differentiable & $\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}}$.

<u>Why?</u> Use implicit differentiation $y = \arctan x$ means $\tan y = x$

$\xrightarrow{\frac{d}{dx}}$ $\sec^2 y \cdot y' = 1$ so $y' = \frac{1}{\sec^2 y}$

But $\sec^2 y = \frac{1}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = 1 + \tan^2 y$

So $y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$ ✓.

<u>Consequence</u> $\boxed{\int \frac{dx}{1+x^2} = \arctan x + C}$

§3 <u>Examples</u>:

① $\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{5}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}} \cdot \frac{1}{5} = \frac{1}{\sqrt{25-x^2}}$.

② $\frac{d}{dx}\left(\sin^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \left(\frac{-1}{x^2}\right) = \frac{-1}{x\sqrt{x^2-1}}$. $\qquad$ (ok $\left|\frac{1}{x}\right| < 1$ so $|x| > 1$)

③ $\frac{d}{dx}\left(\tan^{-1}\left(\sqrt{1+x^2}\right)\right) = \frac{1}{1+(1+x^2)} \cdot \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{(2+x^2)\sqrt{1+x^2}}$

④ $\int \frac{dx}{1+16x^2} \underset{u=4x}{=} \int \frac{1\,du}{4(1+u^2)} = \frac{1}{4}\arctan(u) + C = \frac{1}{4}\arctan(4x) + C$.

⑤ $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{-u^{-1}du}{\sqrt{u^{-2}-1}} = \int \frac{-du}{\sqrt{1-u^2}} = \arcsin(u) + C = \arcsin\frac{1}{x} + C$

$u = \frac{1}{x} \rightsquigarrow x^2 = u^{-2}$

$du = \frac{-dx}{x^2} = -u^2 dx$