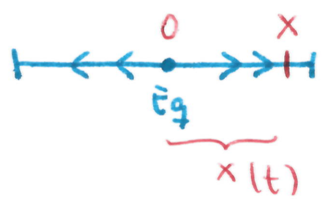
Lecture XXXIV §9.6 Simple Harmonic Motion. The Pendulum

TODAY: Diff'l eqn modelling: · the motions of vibrations producing sound
· oscillations or waves of periodic motions

§1 Simple Harmonic Motion

Def: If an object or point moves back & forth in a straight line (say the x-axis) so that the force required to move it back to equilibrium ($x=0$) is proportional to the distance from equilibrium, we say we have a simple harmonic motion.



$$F(t) = m\frac{d^2x}{dt^2} = -k\,x(t) \quad \text{for } k>0$$

gives $$\frac{d^2x}{dt^2} = -\underbrace{\boxed{\frac{k}{m}}}_{=c>0}\, x(t)$$

Write $c = a^2 > 0$ to emphasize it's positive!

$\rightsquigarrow$ Equation for SHM: $$\boxed{\frac{d^2x}{dt^2} + a^2\,x(t) = 0 \quad \text{for } a>0} \qquad (*)$$

Initial conditions: $\begin{cases} x(t_0) = x_0 & \text{(initial position)} \\ x'(t_0) = v_0 & \text{(initial velocity, Typically } v_0 = 0) \end{cases}$

Ex: $a=1$ gives $x'' + x = 0 \rightsquigarrow$ solns = $\sin(t)$, $\cos(t)$.

In general $x = \alpha \sin(t) + \beta \cos(t)$ for 2 parameters α, β in $\mathbb{R}$ (determined by initial conditions)

Thm: The solutions to SHM $(*)$ are of the form:

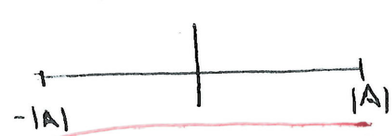
$$x(t) = A\sin(at+b) \quad \text{for some } A, b \text{ in } \mathbb{R}$$

Names: · $|A|$ = amplitude

· $T = \dfrac{2\pi}{a}$ = period $[\,x(t+T) = x(t)$ for all $t.\,]$ = smallest time it takes to return to a position.

$f = \dfrac{1}{T} = \dfrac{a}{2\pi}$ = frequency = number of cycles per second.

Movement oscillated between $-|A|$ & $|A|$
(Largest position = $|A|$, smallest position = $-|A|$)



It takes $\frac{2\pi}{a}$ seconds to travel this cycle

Proof 1: $\frac{d^2x}{dt^2} + a^2 x = 0$ Write $\frac{d^2x}{dt^2} = \frac{dv}{dt}$ & use

Chain Rule $x = x(t)$: $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}\cdot v$ [as in "Escape Velocity"]

(SHM) becomes: $\frac{dv}{dx}\cdot v + a^2 x = 0 \rightsquigarrow v\,dv = -a^2 x\,dx$

Use Separation of variables: $\int v\,dv = \int -a^2 x\,dx$

$$\frac{v^2}{2} = -\frac{a^2}{2}x^2 + C$$

Get $v^2 + a^2x^2 = 2C$ constant. <u>Note</u>: $2C \geq 0$

Use initial conditions $\underbrace{v^2_{(x_0)}}_{v_0} + a^2 x_0^2 = 2C$

So $v^2_{(x)} = v_0^2 + a^2(x_0^2 - x^2) = a^2\Big(\underbrace{\left(\frac{v_0}{a}\right)^2 + x_0^2}_{=:A^2} - x^2\Big)$

Get $\frac{dx}{dt} = v = \pm a\sqrt{A^2 - x^2}$ (sign depends on the direction)

Separation of variables $\int \frac{dx}{\sqrt{A^2-x^2}} = \int \pm a\,dt$

(LHS) $u = \frac{x}{A}$: $\int \frac{dx}{\sqrt{A^2-x^2}} = \frac{1}{A}\int \frac{A\,du}{\sqrt{1-u^2}} = \arcsin(u) = \arcsin\left(\frac{x}{A}\right)$.

(RHS) $\int \pm a\,dt = \pm at + b$

We get $\arcsin\left(\frac{x}{A}\right) = \pm at + b \rightsquigarrow x = A\sin(\pm at + b)$

$\begin{cases} \cdot \text{ If } v > 0: & x = A\sin(at+b) \\ \cdot \text{ If } v < 0: & x = A\sin(-at+b) = (-A)\sin(at-b) \end{cases}$

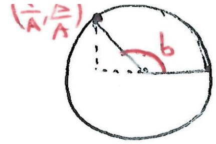
So the signs of A & b depends on the direction of the movement, but the formula look the same. $\square$

<u>Alternative Claim</u>: Any solution has the form $\boxed{x_{(t)} = \alpha\sin(at) + \beta\cos(at)}$. (why? See last page)

. We don't want the trivial solution ($x_{(t)} \equiv 0$), so we can assume either α or $\beta \neq 0$.

Set $A = \sqrt{\alpha^2+\beta^2} > 0$

The point $\left(\frac{\alpha}{A}, \frac{\beta}{A}\right)$ lies in the unit circle



We can find some b with $\frac{\alpha}{A} = \cos b$ & $\frac{\beta}{A} = \sin b$

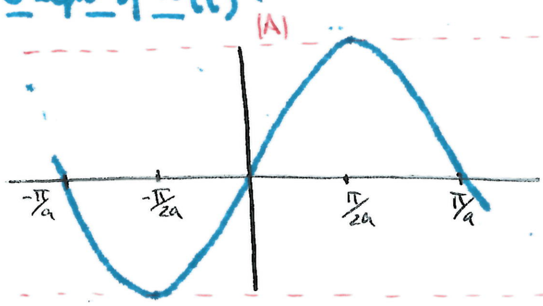
Now $x(t) = \alpha \sin at + \beta \cos at = A \cos b \sin(at) + A \sin b \cos(at)$

$$= A \sin(at + b)$$

last time

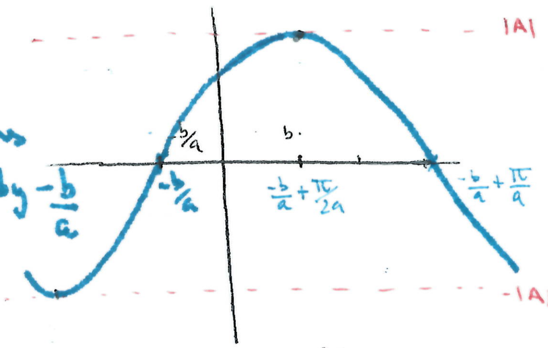
Note: Don't need $A>0$. The expression will also be a solution to (SHM) if $A<0$.

Graph of $x(t)$?



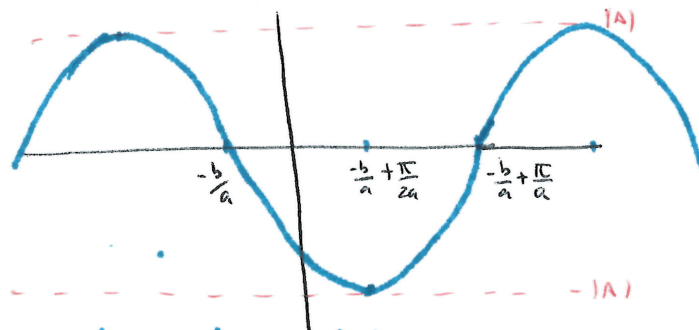
$\boxed{b=0}$ & $\boxed{A>0}$

shift by $-\frac{b}{a}$



ANY b & $A>0$

If $A<0$, rotate about x-axis:

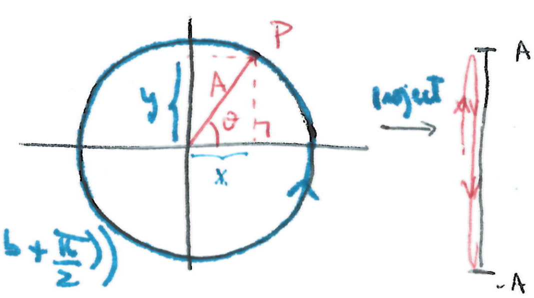


If $A=0$. Soln is $x(t)=0$ (boring!)

Remark $x(t) = A \sin(at+b) = A\cos\left(at + \left(b - \frac{\pi}{2}\right)\right)$

§2 Another perspective

Assume $A>0$. The particle P moves around the circle of radius A with constant angular velocity $\frac{d\theta}{dt} = a \ \frac{\text{rad}}{\text{sec}}$



Soln: $\theta = at + b$ (by integration)

Write: $x = A\cos\theta = A\cos(at+b) = A \sin\left(at + \left(b + \frac{\pi}{2}\right)\right)$

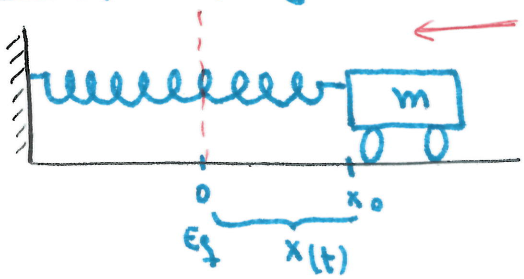
$y = A \sin\theta = \boxed{A\sin(at+b)}$

Projection to the y-axis describes (SHM)

§3 Examples:

(I) SHM for a spring.

Assume: friction & air resistance are negligible.



F restorative force

k = spring constant > 0

$$\boxed{F = -kx}$$

force against movement direction

- Pull cart away from the wall
- Let go of the cart with initial velocity $v_0 = 0$

Newton's Law: $F = m\dfrac{d^2x}{dt^2} = -kx$ gives $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$ (SHM)

Set $a = \sqrt{k/m} > 0$

Soln: $x(t) = \alpha \sin(at) + \beta \cos(at)$

Initial conditions $\begin{cases} x(0) = x_0 \\ \dfrac{dx}{dt}(0) = v_0 \end{cases}$

So $x'(t) = \alpha a \cos(at) - \beta a \sin(at)$ gives $\underset{0}{\underset{\parallel}{v_0}} = x'(0) = \alpha \underset{0}{\underset{\parallel}{a}}$ so $\boxed{\alpha = 0}$

Also $x(0) = 0 + \beta$ gives $\beta = x_0$

Conclusion: $\boxed{x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)} = \overset{A}{x_0} \sin\left(\overset{=a}{\sqrt{\frac{k}{m}}}\, t + \overset{=b}{\frac{\pi}{2}}\right)$

- Amplitude: $x_0 > 0$ (agrees with initial position)
- Period: $T = \dfrac{2\pi}{a} = 2\pi\sqrt{\dfrac{m}{k}}$
- Frequency: $f = \dfrac{1}{T} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$

Observation: • If stiffness k of the spring increases, then the frequency of the vibration increases

• If the mass m increases, then the frequency decreases

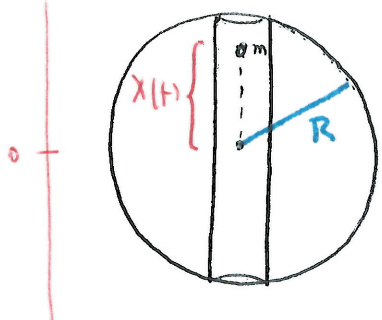
These 2 agree with the expectations from experimental evidence.

(II) Hole in the earth:

Earth = sphere of radius $R = 4000$ mi

- Bore a tunnel straight through the center, from one side to the other
- Drop a body a mass m into the tunnel

GOAL: Motion of the body induced by gravity?



$\downarrow F = -m \dfrac{d^2x}{dt}$

Force of gravity acts as if all the earth were concentrated at the center

Model : $F = -kx$ gives $x'' + \dfrac{k}{m}x = 0$

At the surface $F = -mg$ & $x = R$ so $-mg = -kR$ gives $\boxed{k = \dfrac{mg}{R}}$

Eqn is $\boxed{x'' + \dfrac{g}{R}x = 0}$ (independent of the mass m !)

Conclusion : The object will exhibit SHM (disregard air friction, etc.)

- $a = \sqrt{\dfrac{g}{R}}$
- Period : $\dfrac{2\pi}{a} = 2\pi\sqrt{R/g} \simeq 89$ min $\left(\begin{array}{l} R \approx 8000 \text{ mi} \\ g = 32 \text{ ft/s}^2 \\ 1 \text{ mi} = 5280 \text{ ft} \end{array} \right)$
- Time to get the center $= \dfrac{89}{4} \approx 22$ min
- Amplitude = ? A : Radius of earth

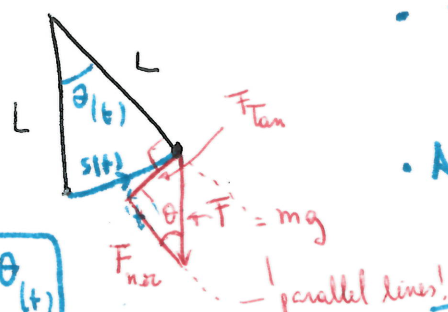
. Why? $x(t) = A\sin(at+b)$ $x(0) = A\sin b = R$

$x'(t) = aA\cos(at+b)$ so $x'(0) = \underbrace{aA}_{\neq 0}\cos b = 0$ so $\cos b = 0$

If $\cos b = 0$, then $\sin b = \pm 1$ so $|A| = \left|\dfrac{R}{\sin b}\right| = \left|\dfrac{R}{\pm 1}\right| = R$.

. Reason 2 : If $v_0 = 0$, amplitude = initial position (same as for spring).

III Pendulum



$\boxed{s(t) = L\theta(t)}$

- Bob (weight) suspended at the end of a light string of length L
- Allowed to swing back and forth under the action of gravity.

$F = mg$, but not in the direction of the movement!

F has 2 components : F_{tan} = tangential to the movement

F_{norm} = perpendicular ———

$\boxed{F_{tan} = -F\sin\theta = -mg\sin\theta}$; $s(t) = L\theta(t)$

Equation: $\frac{d^2 s}{dt^2} = \frac{d^2}{dt^2}(L\theta(t)) = L\frac{d^2\theta}{dt^2} = L\theta''$

Newton's Law $F_{tan} = m\frac{d^2 s}{dt^2} = mL\theta''$

We get: $-mg\sin\theta = mL\theta'' \rightsquigarrow \boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$

For small values of θ: $\sin\theta \approx \theta$ (because $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$) so we get an eqn of SHM $\theta'' + \frac{g}{L}\theta = 0$ with period $= \frac{2\pi}{\sqrt{g/L}} = 2\pi\sqrt{\frac{L}{g}}$

⚠ This is ONLY an approximation! In reality, period depends on the amplitude.
This gives the "circular error" in pendulum clocks.

§4 Alternative solution:

Prop: The general soln to $y'' + a^2 y = 0$ for $a \neq 0$ is $y(t) = \alpha\cos at + \beta\sin at$.

Why? We argue in 4 steps.

STEP 1: If $f(t)$, $g(t)$ are solutions & α, β are real numbers then:

$y = \alpha f(t) + \beta g(t)$ is also a solution.

Pf/ Easy check:

$$
\begin{array}{rl}
y'' = & \alpha f'' + \beta g'' \\
+ \quad a^2 y = & a^2\alpha f + a^2\beta g \\
\hline
y'' + a^2 y = & \alpha(\underbrace{f'' + a^2 f}_{=0}) + \beta(\underbrace{g'' + a^2 g}_{=0}) = \alpha\cdot 0 + \beta\cdot 0 = 0 \checkmark
\end{array}
$$

STEP 2: If $y = f(t)$ is a solution, then $a^2 f(t)^2 + (f'(t))^2$ is constant

Pf/ Take derivative & check it's $= 0$.

$$\frac{d}{dt}\left(a^2 f(t)^2 + (f'(t))^2\right) = a^2\, 2ff' + 2f'f'' = 2f'(\underbrace{a^2 f + f''}_{=0}) = 0 \checkmark$$

STEP 3: If $y(t)$ is a solution & $y(0) = y'(0) = 0$, then $y(t) = 0$ for all t.

Pf/ We use STEP 2: $a^2 y(t)^2 + (y'(t))^2 = \text{const} = a^2(y(0))^2 + (y'(0))^2 = \underset{=0}{0+0}$

So $\underbrace{a^2 y(t)^2}_{\geq 0} + \underbrace{y'(t)^2}_{\geq 0} = 0$ so $a\,y(t) = 0$ & $y'(t) = 0$ are the only options.

Since $a \neq 0$, we get $y(t) = 0$ for all t.

STEP 4 : Write $f(t) = y(t) - \underbrace{\frac{1}{a} y'(0)}_{\in \mathbb{R}} \sin(at) - \underbrace{y(0)}_{\in \mathbb{R}} \cos(at)$

where $y(t)$ is a solution.

. Since $\sin(at)$ & $\cos(at)$ are also solutions, we conclude $f(t)$ is also a soln (use STEP 1)

. Now $f'(t) = y'(0) - y'(0) \cos(at) + a\, y(0) \sin(at)$

In particular: $f(0) = y(0) - 0 - y(0) = 0$

$f'(0) = y'(0) - y'(0) + 0 = 0$

By STEP 3: $f(t) = 0$ for all t

We conclude $y(t) = \underbrace{\boxed{\frac{1}{a} y'(0)}}_{= \beta} \sin(at) + \underbrace{\boxed{y(0)}}_{= \alpha} \cos(at)$ for α, β const.

Exercise: Find the amplitude & frequency of the SHM of a particle with trajectory $x(t) = 3 \sin 2t + 4 \cos 2t$. Find its maximal velocity.

Soln: Want to write $x(t)$ as $A \sin(at + b)$

. Diff'l eqn is $x'' + a^2 x = 0$ $\rightsquigarrow$ $x' = 6 \cos 2t + (-8) \sin 2t$

$x'' = -12 \sin 2t - 16 \cos 2t$

$= -4(3 \sin 2t + 4 \cos 2t) = -4X.$

We get $a^2 = 4$ so take $\boxed{a = 2}$

Period $T = \frac{2\pi}{a} = \pi$ & Frequency $f = \frac{1}{T} = \boxed{\frac{1}{\pi}}$.

. To solve for A & b we need to pick sample pts

- $t = 0$ gives $x(0) = 3 \cdot 0 + 4 \cos 0 = 4$

  $x(0) = A \sin b$

- $t = \frac{\pi}{4}$ gives $x(\frac{\pi}{4}) = 3 \sin \frac{\pi}{2} + 4 \cos \frac{\pi}{2} = 3 + 4 \cdot 0 = 3$

  $x(\pi/4) = A \sin(\frac{\pi}{2} + b) = A \cos b$

So $\left.\begin{array}{l} A \sin b = 4 \\ A \cos b = 3 \end{array}\right\}$ $A^2(\underbrace{\sin^2 b + \cos^2 b}_{= 1}) = 16 + 9 = 25$ so $A = \pm 5$

Amplitude $= |A| = 5$.

- Velocity: $x'(t) = 6 \cos 2t - 8 \sin 2t = 2 \overset{\pm 5}{A} \sin(2t + b)$ $\rightsquigarrow$ max'l value $= 10 = 2|A|$.