Lecture XXXV: §10.1 The basic formulas
§10.2 The method of substitution
§10.5 Certain trigonometric integrals

§1 The Basic formulas

Def: An elementary function is one built from x^a, e^x, $\ln(x)$, $\sin(x)$, $\cos(x)$, $\sin^{-1}(x)$ & $\tan^{-1}(x)$, & constants. [add, multiply, divide, scaler mult & compose]

Ex: $\tan^{-1}\left(\dfrac{\ln(x^2+\cos^2(x))}{e^x+\sin\sqrt{x^{2.5}+1}}\right)+1.$

Note
- Simple rules for differentiation of building blocks + Chain / Prod Rules gives easy way to differentiate elem. functions
- Integration is more subtle: no systematic way & answer need NOT be an elementary function. We have a recognition problem (what method to use, how to apply it?)

Example. $Li(x) = \displaystyle\int_2^x \frac{dt}{\ln t}$; $\displaystyle\int_0^x e^{-t^2}\,dt$ are not expressible as elem. functions.
[Appendix A9]

- 15 basic formulas in handout, pages 335 & 336 of Textbook.

§2 Method of substitution

- Substitution is the analogue of the Chain Rule for integration ⇝ recognition problem!

Substitution Rule: $\displaystyle\int_a^t f'(g(x))\,g'(x)\,dx \underset{\substack{u=g(x)\\du=g'(x)dx}}{=} \int_{g(a)}^{g(t)} f'(u)\,du \underset{FTC}{=} f(u)\Big|_{g(a)}^{g(t)} = f(g(t)) - f(g(a))$

Examples: ① $\displaystyle\int x e^{-x^2}dx \underset{\substack{u=-x^2\\du=-2x\,dx}}{=} \int e^u \frac{du}{-2} = -\frac{e^u}{2}+C = -\frac{e^{-x^2}}{2}+C.$

② $\displaystyle\int \frac{\cos(x)\,dx}{\sqrt{1+\sin(x)}} \underset{\substack{u=1+\sin x\\du=\cos x\,dx}}{=} \int u^{-1/2}du = 2u^{1/2}+C = 2\sqrt{1+\sin x}+C$

Note. Subst $u=\cos x$ or $u=\sqrt{1+\sin x}$ won't work.

③ $\displaystyle\int \frac{dx}{x\ln x} \underset{\substack{u=\ln x\\du=dx/x}}{=} \int\frac{du}{u} = \ln u + C = \ln(\ln x)+C$

④ $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{3\sqrt{1-\frac{4}{9}x^2}} \underset{\substack{u=\frac{2}{3}x \\ du=\frac{2}{3}dx}}{=} \frac{1}{3}\int \frac{\frac{3}{2}du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin\left(\frac{2}{3}x\right)+C$

⑤ $\displaystyle\int \frac{x\,dx}{\sqrt{9-4x^2}} = \frac{1}{3}\int \frac{x\,dx}{\sqrt{1-\frac{4}{9}x^2}} \underset{\substack{u=1-\frac{4}{9}x^2 \\ du=-\frac{8}{9}x\,dx}}{=} \frac{1}{3}\int \frac{-\frac{9}{8}du}{\sqrt{u}} = \frac{-3}{8}\frac{u^{1/2}}{1/2}+C$

$\displaystyle = -\frac{3}{4}\sqrt{1-\frac{4}{9}x^2}+C$

§10.3 Certain Trigonometric functions:

GOAL: Find a method for integrating 3 functions

① $\int \sin^m(x)\cos^n(x)\,dx$

② $\int \tan^m(x)\sec^n(x)\,dx$ for $m, n \geq 0$ integers

③ $\int \cot^m(x)\csc^n(x)\,dx$

Why? $\frac{d\sin x}{dx} = \cos x$, $\frac{d\tan x}{dx} = \sec^2(x)$, $\frac{d\cot(x)}{dx} = -\csc^2(x)$.

⚠ Answer will depend on the parity of m & n

Examples

① $(n=1)$ $\displaystyle\int \sin^m(x)\cos x\,dx \underset{u=\sin x}{=} \int u^m\,du = \frac{\sin^{m+1}(x)}{m+1}+C$

② $(n=2)$ $\displaystyle\int \tan^m(x)\sec^2(x)\,dx \underset{u=\tan x}{=} \int u^m\,du = \frac{\tan^{m+1}(x)}{m+1}+C$

③ $(n=2)$ $\displaystyle\int \cot^m(x)\csc^2(x)\,dx \underset{u=\cot(x)}{=} -\frac{\cot^{m+1}(x)}{m+1}+C$

Q: What about other values of n?

A: Do it for ①, others are similar. We separate in 2 cases (~~even~~/odd comb.)

CASE A: m or n are ODD (eg: 1, 3, 5, ...)

Trick: Use trig identities to turn integrand into one of 2 options:

(i) $\cos^a(x)\sin(x)$ (for m ODD) or (ii) $\sin^a(x)\cos x$ (for n ODD)

(ii) For n ODD, write $n = 2k+1$ for some $k \geq 0$ integer. Then.

$$\boxed{\cos^{2k+1}(x)} = \cos^{2k}(x)\cos(x) = (\cos^2(x))^k \cos(x) = \boxed{(1-\sin^2 x)^k \cos x}$$

The Binomial Thm gives the answer:

$$(1-\sin^2 x)^k = 1 - k\sin^2(x) + \binom{k}{2}\sin^4 x - \cdots + (-1)^k \sin^{2k}(x)$$

(only even powers of $\sin x$).

$$\text{So}\quad \sin^m(x)\cos^{2k+1}(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \underbrace{\sin^{2j+m}(x)\cos x}$$

↳ can integrate with substitution $u = \sin(x)$

(i) For m ODD, reverse the roles of $\sin(x)$ & $\cos(x)$.

$$\sin^{2k+1}(x)\cos^n(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \underbrace{\cos^{2j+n}(x)\sin x}$$

↳ can integrate with subs $u = \cos(x)$

Examples (i) $[m=3, n=0]$ $\int \sin^3 x\, dx = \int \sin x \sin^2 x\, dx = \int \sin x(1-\cos^2 x)\, dx$

$$= \int \sin x\, dx - \int \sin x \cos^2 x\, dx \underset{u=\cos x}{=} -\cos x + \frac{\cos^3 x}{3} + C$$

(ii) $[m=2, n=5]$ $\int \sin^2(x)\cos^5 x\, dx = \int \sin^2 x\,(1-\sin^2 x)^2 \cos x\, dx$

$$= \int \sin^2 x\,(1+\sin^4(x) - 2\sin^2 x)\cos x\, dx = \int \sin^2 x\cos x\, dx + \int \sin^6 x\cos x\, dx$$

$$- 2\int \sin^4 x\cos x\, dx \underset{u=\sin x}{=} \frac{\sin^3 x}{3} + \frac{\sin^7(x)}{7} - \frac{2}{5}\sin^5 x + C$$

CASE B: Both m & n are even $(0, 2, 4, \ldots)$

- Write $m = 2k$ & $n = 2l$ for $k, l \geq 0$ integers
- Use half-angle formulas!

$$\begin{cases} \cos^2 x + \sin^2 x = 1 \\ \cos^2 x - \sin^2 x = \cos(2x) \end{cases} \quad \text{gives} \quad \boxed{\begin{aligned} 2\cos^2 x &= 1 + \cos(2x) \\ 2\sin^2 x &= 1 - \cos(2x) \end{aligned}} \quad \begin{aligned} &\text{(add 2 eqns)} \\ &\text{(subtr. 2 eqns)} \end{aligned}$$

Idea: $\cos^{2k}(x)\sin^{2l}(x) = \left(\frac{1+\cos 2x}{2}\right)^k \left(\frac{1-\cos 2x}{2}\right)^l$

- Open up, use ½ angle formulas until one of the exponents becomes ~~odd~~. Then use Case A.

Example $\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx \overset{\frac{1}{2}\text{ angle}}{=} \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$\underset{\text{Binomial}}{=} \frac{1}{4} \int 1 + 2\cos 2x + \cos^2 2x \, dx = \frac{1}{4}\left(x + \operatorname{sen} 2x + \int \cos^2(2x)\,dx\right)$

$\int \cos^2 2x \, dx \underset{u=2x}{=} \int \cos^2 u \frac{du}{2} \overset{\frac{1}{2}\text{ angle}}{=} \int \left(\frac{1+\cos 2u}{2}\right)\frac{du}{2} = \frac{1}{4}\int (1+\cos 2u)\,du$

$= \frac{1}{4}\left(u + \frac{\operatorname{sen} 2u}{2}\right) = \frac{1}{4}\left(2x + \frac{\operatorname{sen} 4x}{2}\right)$

So $\int \cos^4 x \, dx = \frac{1}{4}\left(x + \operatorname{sen} 2x + \frac{x}{2} + \frac{\operatorname{sen} 4x}{8}\right) + C$

$= \frac{3x}{8} + \frac{\operatorname{sen} 2x}{4} + \frac{\operatorname{sen} 4x}{32} + C$

What about ② = $\int \tan^m(x) \sec^n(x)\,dx$? And ③ = $\int \cot^m(x) \csc^n(x)$?

3 cases:
- CASE A = m odd
- CASE B = n even

} overlap (if m odd & n even, use Ⓐ or Ⓑ)

- CASE C : neither, so m even & n odd

. Use trig identities : ② $\begin{cases} 1+\tan^2 x = \sec^2(x) & (B) \\ \tan^2 x = (\sec^2 x - 1) & (A) \end{cases}$ ③ $\begin{cases} 1+\cot^2 x = \csc^2 x & (B) \\ \cot^2 x = \csc^2(x) - 1 & (A) \end{cases}$

. Use substitutions ② $\begin{cases} d\tan x = \sec^2 x \, dx \\ d\sec x = \sec x \tan x \, dx & (A) \end{cases}$ ③ $\begin{cases} d\cot x = -\csc^2 x \, dx \\ d\csc x = -\cot x \csc x \, dx \end{cases}$

. Binomial Thm

Examples for ②:

Ⓐ $[m=3, n=1]$ $\int \tan^3 x \sec(x)\,dx \overset{\boxed{\text{pull out 1 Tan x}}}{=} \int \tan x \,(\tan^2 x) \sec(x)\,dx$

$\underset{\text{subs.}}{=} \int \tan x\,(\sec^2 x - 1)\sec x \, dx \underset{\boxed{\text{group Tan x sec x}}}{=} \int \underbrace{(\tan x \sec x)}_{d\sec x} \sec^2 x \, dx - \int \underbrace{\tan x \sec x \, dx}_{d(\sec x)}$

$= \frac{\sec^3 x}{3} - \sec(x) + C$

Steps
1. $\tan^{2k+1}(x) \sec x = \tan x \sec x \,(\sec^2 x - 1)^k$
2. Expand $(\sec^2 x - 1)^k$ with binomial coeff.
3. Use $u = \sec x$ a substitution since $du = \tan x \sec x \, dx$.

(b) $[m=4,\ n=6]$ $\displaystyle\int \tan^4 x \sec^6 x\,dx \underset{\text{pull } \sec^2 x \text{ out}}{=} \int \tan^4 x \sec^4 x \boxed{\sec^2 x}\,dx$

$$\underset{\text{replace } \sec^4 x}{=} \int \tan^4 x\,(1+\tan^2 x)^2 \sec^2 x\,dx \underset{\text{Binomial}}{=} \int \tan^4 x\,(1+\tan^4 x+2\tan^2 x)\sec^2 x\,dx$$

$$= \int \tan^4 x \underbrace{\sec^2 x\,dx}_{=d\tan x} + \int \tan^8 x \underbrace{\sec^2 x\,dx}_{d(\tan x)} + 2\int \tan^6 x \underbrace{\sec^2 x\,dx}_{d(\tan)}$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^9 x}{9} + \frac{2\tan^7 x}{7} + C$$

Steps: 1. $\tan^m x \sec^{2k} x\,dx = \tan^m x \sec^{2(k-1)} x \sec^2 x\,dx$

2. Write $\sec^{2(k-1)} x = (1+\tan^2 x)^{k-1}$ & expand with binomial coeff

3. Use $u=\tan x$ substitution.

(c) Don't know how to solve for now, with one exception $\int \sec x\,dx$ in handout
$m=0,\ n=1.$

Remark: Sometimes it's easier to express (2) & (3) via $\sin x$ & $\cos x$.

Example $\displaystyle\underbrace{\int \tan x \cdot \sec x\,dx}_{} = \int \frac{\sin x}{\cos x}\frac{1}{\cos x}\,dx = \int \frac{\sin x}{\cos^2 x}\,dx$

$$\underset{u=\cos x}{=} \int \frac{-du}{u^2} = u^{-1} + C = \frac{1}{\cos x} + C = \underbrace{\sec x}_{} + C$$

(we already knew this!)

Integration formulas

[C = constant]

① $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C$ for $n \neq 1$

② $\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln u + C$

③ $\int e^u\,du = e^u + C$

④ $\int \cos u\,du = \operatorname{sen} u + C$

⑤ $\int \operatorname{sen} u\,du = -\cos u + C$

⑥ $\int \sec^2 u\,du = \tan u + C$

⑦ $\int \csc^2 u\,du = -\operatorname{cotan} u + C$

⑧ $\int \sec u \tan u\,du = \sec u + C$

⑨ $\int \csc u \operatorname{cotan} u\,du = -\csc u + C$

⑩ $\int \dfrac{du}{\sqrt{a^2-u^2}} = \operatorname{sen}^{-1}\left(\frac{u}{a}\right) + C$ ($x = \frac{u}{a}$ substitution)

⑪ $\int \dfrac{du}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ ($x = \frac{u}{a}$ ——)

⑫ $\int \tan(u)\,du = -\ln \cos u + C$ ($x = \cos u$ ——)

⑬ $\int \cot(u)\,du = \ln \operatorname{sen}(u) + C$ ($x = \operatorname{sen} u$ ——)

⑭ $\int \sec(u)\,du = \ln\left(\sec(u) + \tan(u)\right) + C$

⑮ $\int \csc(x)\,du = -\ln\left(\csc(u) + \cot(u)\right) + C$