Lecture XXXVI §10.4 Trigonometric Substitution
§10.5 Completing the square

§1 Trigonometric substitutions

. Used to compute integrals involving 3 expressions:

$$\boxed{(1)\ \sqrt{a^2-x^2} \qquad (2)\ \sqrt{a^2+x^2} \qquad (3)\ \sqrt{x^2-a^2}}$$

where $a>0$ is constant.

. For each we use a different substitution of x as a Trig function.

(1) $x = a\,\text{sen}(u) \rightsquigarrow \sqrt{a^2-x^2} = \sqrt{a^2(1-\text{sen}^2 u)} = a\cos(u)$
$\boxed{\cos^2 u + \text{sen}^2 u = 1}$

(2) $x = a\tan(u) \rightsquigarrow \sqrt{a^2+x^2} = \sqrt{a^2(1+\tan^2 u)} = \sqrt{a^2\sec^2 u} = a\sec(u)$
$\boxed{1+\tan^2 u = \sec^2 u}$

(3) $x = a\sec(u) \rightsquigarrow \sqrt{x^2-a^2} = \sqrt{a^2(\sec^2 u - 1)} = \sqrt{a^2\tan^2 u} = a\tan u$

Saw (1) & (2) when we did inverse Trig functions ($\arcsin(x)$ & $\arctan(x)$).

Geometric interpretation = after examples.

3 Examples:

(1) $$\int \frac{\sqrt{a^2-x^2}}{x}\,dx \underset{\substack{x = a\,\text{sen}\,u \\ dx = a\cos u\,du}}{=} \int \frac{a\cos u}{a\,\text{sen}\,u}\,a\cos u\,du = a\int\frac{\cos^2 u}{\text{sen}\,u}\,du = a\int\frac{1-\text{sen}^2 u}{\text{sen}\,u}\,du$$

$$= a\int \underbrace{\frac{1}{\text{sen}\,u}}_{\csc(u)}\,du - a\int \text{sen}\,u\,du \underset{\text{Table}}{=} -a\ln(\csc(u)+\cot(u)) + a\cos u + C$$

In terms of x? $\csc(u) = \dfrac{1}{\text{sen}\,u} = \dfrac{a}{x}$; $\cot(u) = \dfrac{\cos u}{\text{sen}\,u} = \dfrac{\sqrt{1-\text{sen}^2 u}}{x/a} = \dfrac{\sqrt{1-x^2/a^2}}{x/a} = \dfrac{\sqrt{a^2-x^2}}{x}$

$\cos(u) = \sqrt{1-\text{sen}^2 u} = \sqrt{1-\dfrac{x^2}{a^2}} = \dfrac{\sqrt{a^2-x^2}}{a}$

Conclusion: $$\int\frac{\sqrt{a^2-x^2}}{x}\,dx = -a\ln\left(\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}\right) + \sqrt{a^2-x^2} + C$$

(2) $$\int\frac{dx}{x^2\sqrt{a^2+x^2}} \underset{\substack{x = a\tan u \\ dx = a\sec^2 u\,du}}{=} \int\frac{a\sec^2 u\,du}{a^2\tan^2 u\; a\sec u} = \frac{1}{a^2}\int\frac{\sec u}{\tan^2 u}\,du = \frac{1}{a^2}\int\frac{\cos^2 u}{\text{sen}^2 u}\frac{du}{\cos u}$$

$$= \frac{1}{a^2}\int\frac{\cos u\,du}{\text{sen}^2 u} \underset{y = \text{sen}\,u}{=} \frac{1}{a^2}\int y^{-2}\,dy = \frac{-1}{a^2 y} + C = \boxed{\frac{-1}{a^2\,\text{sen}\,u} + C}$$

Again, write it in terms of x:

$$\underset{\frac{1}{\cos u}}{a \sec u} = \sqrt{a^2+x^2} \quad \text{gives} \quad \cos u = \frac{a}{\sqrt{a^2+x^2}}$$

$$\text{So } \sin u = \sqrt{1-\cos^2 u} = \sqrt{1-\frac{a^2}{a^2+x^2}} = \frac{x}{\sqrt{a^2+x^2}}$$

Conclusion: $\displaystyle\int \frac{dx}{x^2\sqrt{a+x^2}} = \frac{-\sqrt{a^2+x^2}}{a^2 x} + C$

Verification: $\displaystyle\frac{d}{dx}\left(\frac{-\sqrt{a^2+x^2}}{a^2 x}\right) = \frac{\frac{-x a^2 x}{\sqrt{a^2+x^2}} + a^2\sqrt{a^2+x^2}}{a^4 x^2} = \frac{-x^2+a^2+x^2}{a^2 x^2\sqrt{a^2+x^2}}$

$$= \frac{1}{x^2\sqrt{a^2+x^2}} \checkmark$$

(3) $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx \underset{\substack{x=a\sec(u)\\ dx=a\sec(u)\tan u\,du}}{=} \int \frac{a\tan u}{a\sec u}\, a\sec u \tan u\, du = \int a\tan^2 u\, du$

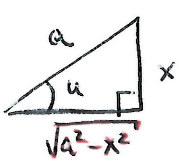
$$\underset{\text{Last time}}{=} a\int(\sec^2 u-1)\,du = a\int \underbrace{\sec^2 u\,du}_{=d\tan u} - a\int du = a\tan u - au + C$$

Again, write it in terms of x:

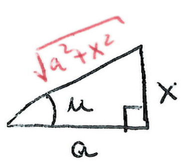
$$a\tan u = \sqrt{x^2-a^2}, \quad u = \tan^{-1}\left(\frac{\sqrt{x^2-a^2}}{a}\right)$$

Conclusion: $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx = a^2\sqrt{x^2-a^2} - a\tan^{-1}\left(\frac{\sqrt{x^2-a^2}}{a}\right) + C$

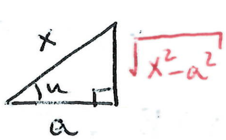
Geometric interpretation: sides of a right triangle. (2 given & complete 3rd one)

(1) 

$\boxed{x = a\sin u} \rightsquigarrow \cos u = \frac{\sqrt{a^2-x^2}}{a}$ & $\tan u = \frac{x}{\sqrt{a^2-x^2}}$

(2) 

$\boxed{x = a\tan u} \rightsquigarrow \cos u = \frac{a}{\sqrt{a^2+x^2}}$ & $\sin u = \frac{x}{\sqrt{a^2+x^2}}$

(3) 

$a = x\cos u$

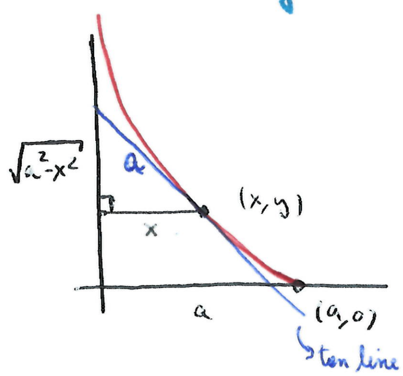
$\boxed{x = \frac{a}{\cos u} = a\sec u} \rightsquigarrow \cos u = \frac{a}{x}$; $\sin u = \frac{\sqrt{x^2-a^2}}{x}$

; $\tan u = \frac{\sqrt{x^2-a^2}}{a}$

Application: tractrix curve

Path of an object dragged along a horizontal plane by a string of fixed length when the other end of the string moves along a straight line in the plane

- Fix string of length $a>0$
- Start w/ object on $(a,0)$ & other end of string at $(0,0)$
- Move one end up along y-axis & other along the tractrix (to the left).
- The string is always tangent to the curve.



Slope of tangent $= \dfrac{dy}{dx} = \dfrac{\sqrt{a^2-x^2}}{-x}$

Sep of variables gives $\quad dy = -\dfrac{\sqrt{a^2-x^2}}{x}\,dx$

$$y = \int dy = -\int \frac{\sqrt{a^2-x^2}}{x}\,dx \underset{\text{Example 1}}{=} a\ln\left(\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}\right) - \sqrt{a^2-x^2} + C$$

But $y(a)=0$ so $\quad 0 = a\ln\left(\frac{a}{a}+0\right) - 0 + C = a\ln 1 + C = C$

Eqn of tractrix $= \boxed{y = a\ln\left(\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}\right) - \sqrt{a^2-x^2} \quad (0<x\leq a)}$

§2. Completing the square:

Why? Want to integrate expressions involving

(1) $ax^2+bx+c \quad$ & $\quad$ (2) $\sqrt{ax^2+bx+c}$

for a, b, c fixed constants, $a\neq 0$.

To use trig subst, need to write this as $\pm(A^2 \pm u^2)$

IDEA: Reverse process of squaring a sum.

$$(x+A)^2 = x^2 + 2Ax + A^2$$

Example 1: $5+4x-x^2 = -(x^2-4x-5) = -(x^2-4x+\underbrace{4}_{(4/2)^2}-4-5)$

$$= -((x-2)^2-9) = 9-(x-2)^2$$

So $\displaystyle\int \frac{x+2}{\sqrt{5+4x-x^2}}\,dx = \int \frac{x+2}{\sqrt{9-(x-2)^2}}\,dx \underset{u=x-2}{=} \int \frac{u+4}{\sqrt{9-u^2}}\,du \rightsquigarrow$ Trig subs. (1) $a=3$, $u = 3\sin y$

Example 2 $\displaystyle\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{9+(x+1)^2} \underset{u=x+1}{=} \int \frac{du}{9+u^2} \underset{u=\frac{v}{3}}{=} \frac{1}{3}\int \frac{1}{3}\frac{dv}{1+v^2} = \frac{1}{9}\arctan(v) + C$

$x^2+2x+10 = x^2+2x+1-1+10 = (x+1)^2+9$

$= \frac{1}{9} \arctan\left(\frac{x+1}{3}\right) + C$

Example 3 $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-2x+5}} = \int \frac{x\,dx}{\sqrt{4+(x-1)^2}} \underset{u=x-1}{=} \int \frac{u+1}{\sqrt{4+u^2}}\,du = \int \frac{u\,du}{\sqrt{4+u^2}} + \int \frac{du}{\sqrt{4+u^2}}$

$x^2-2x+5 = x^2-2x+1-1+5 = (x-1)^2+4$

Trig subs (2)
$a = 2$
$u = a\tan(y)$

Example 4 $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+3}} = \int \frac{dx}{\sqrt{(x-2)^2-1}} \underset{u=x-2}{=} \int \frac{du}{\sqrt{u^2-1}} \rightsquigarrow$ Trig subs (3)
$a = 1$
$u = a\sec(y)$

$x^2-4x+3 = x^2-4x+4-4+3 = (x-2)^2-1$

In general: $a \neq 0$

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$
$$= a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$$
$$= a\left(\left(x+\frac{b}{2a}\right)^2+\frac{4c-ab^2}{4a^2}\right)$$

The sign of $4c-ab^2$ determines which substitution to do.

3 options >0, <0, or $=0$