Lecture XXXVII : §10.6 Partial Fractions

GOAL: Integrate $\int \boxed{\frac{P(x)}{Q(x)}} dx$ where P, Q are polynomials with real coeffs.

→ Name: Rational Function

§1 Long division of polynomials

- Input: P, Q.
- Output: $\underset{\text{quotient}}{P_1(x)}$ & $\underset{\text{remainder}}{R(x)}$ with $R=0$ or $\deg R(x) < \deg Q$

satisfying $P(x) = P_1(x) Q(x) + R(x)$

Long division by example

$$P = x^3 - 3x^2 \quad \big\lfloor\, x^2+1 = Q$$

$$\begin{array}{l} x^3 - 3x^2 \\ -\ (x^3 + x) \\ \hline -3x^2 - x \\ -\ (-3x^2 - 3) \\ \hline 0\ \boxed{-x+3} = R \end{array} \qquad \boxed{x-3} = P_1$$

→ degree $=1<2=\deg Q$

So $x^3-3x^2 = \underset{P_1}{(x-3)}(x^2+1) + \underset{R}{(-x+3)}$

Routine

1. Take leading terms (highest deg monomial + coefficient) & write ratio ($x = \frac{x^3}{x^2}$; $-3 = \frac{-3x^2}{x^2}$)
2. Multiply ratio by Q & substract it from P
3. Repeat with new polynomial until we get 0 or degree $< \deg(Q)$
4. Left-over is the remainder R
   - Sum of all ratios is $P_1(x)$.

Example 2:

$$\begin{array}{l} x^5 + 2x + 1 \\ -\ (x^5 - 2x^2) \\ \hline 2x^2+2x+1 \end{array} \quad \big\lfloor\, x^3-2 \qquad x^2$$

$\rightsquigarrow x^5+2x+1 = x^2(x^3-2) + (2x^2+2x+1)$

So $\frac{x^5+2x+1}{x^3-2} = x^2 + \frac{2x^2+2x+1}{x^3-2}$

Conclusion: Using long division, we've reduced the integration problem to

$$\int \frac{R(x)}{Q(x)}\, dx \quad \text{with} \quad \deg R < \deg Q.$$

FACT: Every polynomial over $\mathbb{R}$ is a product of $\underset{(\deg 1)}{\text{linear}}$ & irreducible deg 2 polynomials in $\mathbb{R}$ ↳ no real roots

- We'll use this to simplify our task: Q will be of the form $(x-\lambda)^m$ or $(x^2+bx+c)^m$

Q How? A: Partial fractions!

§2 Partial fractions

IDEA: Write $\frac{R(x)}{Q(x)}$ with $\deg R < \deg Q$ as a sum of rational functions of the form $\frac{1}{(x-a)^m}$ & $\frac{Ax+B}{(x^2+bx+c)^r}$ with $m, r > 0$ integers

How? . Reverse the process of getting common denominators
. Denominators come from factorizing Q.

. Example 1: $\frac{12x-7}{(x-1)(x-2)} = \frac{-5}{x-1} + \frac{17}{x-2}$ (−5, 17 need to be computed)

So $\int \frac{12x-7}{(x-1)(x-2)}\,dx = -5\ln|x-1| + 17\ln|x-2| + C$

. Several cases to analyze, depending on factors of $Q(x)$ & their multiplicities

$$\boxed{\text{CASE 1: } Q(x) = (x-r_1)(x-r_2)\cdots(x-r_n) \quad \text{all real roots \& all distinct}}$$

We write $\frac{R(x)}{Q(x)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n}$

We find $A_1, \ldots, A_n$ in 2 ways:

(1) Evaluating at n random numbers $(\neq r_1, \ldots, r_n)$: n eqns in n unknowns

(2) Take common denominator & equate each coefficient in $R(x)$ to that on the (RHS)

Ex 2: $\frac{9x^2+6}{x(x-2)(x-3)} = \frac{A_1}{x} + \frac{A_2}{x-2} + \frac{A_3}{x-3}$

Method 1:

$$\left.\begin{aligned}
x=1:&\quad \frac{15}{1(-1)(-2)} = A_1 - A_2 - \frac{A_3}{2}\\
x=-1:&\quad \frac{15}{-1(-3)(-4)} = -A_1 - \frac{A_2}{3} - \frac{A_3}{4}\\
x=-2:&\quad \frac{15}{(-2)(-4)(-5)} = \frac{A_1}{-2} - \frac{A_2}{4} - \frac{A_3}{5}
\end{aligned}\right\}$$

3 linear equations in A_1, A_2, A_3

$$\begin{cases} 15 = 2A_1 - 2A_2 - A_3 \\ 15 = 12A_1 + 4A_2 + 3A_3 \\ 15 = 20A_1 + 10A_2 + 8A_3 \end{cases}$$

Method 2: Common Denominator

$$\begin{aligned}\text{Num} &= A_1(x-2)(x-3) + A_2\,x(x-3) + A_3\,x(x-2) \qquad (*)\\
&= A_1(x^2-5x+6) + A_2(x^2-3x) + A_3(x^2-2x)\end{aligned}$$

$$= (A_1 + A_2 + A_3)\, x^2 + (-5A_1 - 3A_2 - 2A_3)\, x + 6A_1 \overset{?}{=} R_{(x)} = 9x^2 + 6$$

3 eqns: $A_1 + A_2 + A_3 = 9$ (coeff x^2)

$-5A_1 - 3A_2 - 2A_3 = 0$ (— x)

$6A_1 = 6$ (— 1) $\rightsquigarrow \boxed{A_1 = 1}$

replace back in previous 2 eqns.

$\begin{cases} 1 + A_2 + A_3 = 9 \\ -5 - 3A_2 - 2A_3 = 0 \end{cases}$

So $A_2 + A_3 = 8 \rightsquigarrow A_2 = 8 - A_3$

$-3A_2 - 2A_3 = 5$

replace

$-3(8 - A_3) - 2A_3 = 5$

$-24 + A_3 = 5$

$\boxed{A_3 = 29}$ so $A_2 = 8 - 29 = -21$

Conclude: $\boxed{A_3 = 29, \quad A_2 = -21, \quad A_1 = 1}$

Alternative way: $A_1(x-2)(x-3) + A_2\, x(x-3) + A_3\, x(x-2) \overset{?}{=} R(x)$

Write numerator & evaluate at $x = r_1, \dots, r_n$ $[0, 2, 3]$. All but one summand survives each time.

$$\begin{cases} x = 0 & A_1(-2)(-3) + 0 + 0 = 9 \cdot 0 + 6 \rightsquigarrow \boxed{A_1 = 1} \\ x = 2 & 0 + A_2\, 2(-1) + 0 = 9 \cdot 4 + 6 = 42 \rightsquigarrow \boxed{A_2 = -21} \\ x = 3 & 0 + 0 + A_3\, 3(1) = 9 \cdot 9 + 6 = 87 \rightsquigarrow \boxed{A_3 = 29} \end{cases}$$

Answer: $\displaystyle\int \frac{9x^2 + 6}{x(x-2)(x-3)}\, dx = \int \frac{1}{x} \; \frac{-21}{x-2} + \frac{29}{x-3}\, dx = \ln|x| - 21 \ln|x-2| + 29 \ln|x-3| + C.$

CASE 2: All real roots but with multiplicities, ie $(x-r)^m$ shows up in factorization of Q & $m \geq 2$

Soln: Replace $\dfrac{A_k}{x - r_k}$ from case 1 by the sum $\dfrac{A_{k,1}}{(x-r_k)} + \dfrac{A_{k,2}}{(x-r_k)^2} + \dots + \dfrac{A_{k,m}}{(x-r_k)^m}$

where $m = \text{mult}(r_k, Q)$. [as many summands as multiplicity!]

We find $A_{k,1}, \dots, A_{k,m}$ as we did in case 1.

Ex 3: $\dfrac{2x+1}{(x-1)^3} = \dfrac{A_1}{x-1} + \dfrac{A_2}{(x-1)^2} + \dfrac{A_3}{(x-1)^3}$

Numerator: $2x + 1 \overset{?}{=} A_1 (x-1)^2 + A_2 (x-1) + A_3$

- Evaluate at $x = 1$, all but 1 summand survives (A_3 gets fixed)
- Replace A_3, factor out $x-1$ & evaluate at $x=1$ to get A_2, repeat

- $x=1 \rightsquigarrow 2x+1 = 0 + 0 + A_3 \rightsquigarrow \boxed{A_3=3}$ ($3 = 2\cdot 1+1$)

$2x-2 = 2x+1-3 = A_1(x-1)^2 + A_2(x-1) \underset{\substack{\text{Divide by}\\ (x-1) \text{ on both sides}}}{\rightsquigarrow} 2 = A_1(x-2) + A_2$

$= 2(x-1)$

- $\underline{x=1}$ gives $\boxed{2 = A_2}$

Substitute: $2-2 = 0 = A_1(x-2)$ gives $\boxed{A_1 = 0}$

Alternative way: Evaluate at $x=1$; Take derivatives up to order $m-1$ & evaluate all at $x=1$

$$2x+1 = A_1(x-1)^2 + A_2(x-1) + A_3 \underset{x=1}{\rightsquigarrow} 3 = A_3$$
$$2 = 2A_1(x-1) + A_2 \underset{x=1}{\rightsquigarrow} 2 = A_2$$
$$0 = 2A_1 \underset{x=1}{\rightsquigarrow} 0 = 2A_1 \text{ so } A_1 = 0.$$

Answer: $\displaystyle\int \frac{2x+1}{(x-1)^3}\,dx = \int \frac{2}{(x-1)^2} + \frac{3}{(x-1)^3}\,dx = \frac{-2}{(x-1)} - \frac{3}{2(x-1)^2} + C$

CASE (3): We have quadratic factors (without real roots): x^2+bx+c

- Decomposition depends on multiplicity of the factor.
- If $\underline{\text{mult}=1}$, Then we have a summand $\dfrac{Ax+B}{x^2+bx+c}$ in partial fraction
- If $\underline{\text{mult}=m>1}$ ——— m summands in partial fraction, namely

$$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(x^2+bx+c)^m}$$

- We find the values of (A,B), resp $(A_1, B_1, \ldots, A_m, B_m)$ with the same methods as in Cases 1 & 2 above

Ex 4 $\displaystyle\int \frac{x^2+2}{x^4+2x^3+2x^2}\,dx = \int \frac{x^2+2}{x^2(x^2+2x+2)}\,dx$

Roots of x^2+2x+2? Quadratic formula gives $\dfrac{-2 \pm \sqrt{\overset{<0}{4-8}}}{2}$ so no real roots!

Write $\dfrac{x^2+2}{x^2(x^2+2x+2)} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dfrac{A_3x+B_3}{x^2+2x+2}$

Numerator: $x^2+2 = x(x^2+2x+2)A_1 + A_2(x^2+2x+2) + x^2(A_3x+B_3)$

Evaluate at $x=0$: $2 = 0 + A_2 \cdot 2 + 0 \rightsquigarrow \boxed{A_2 = 1}$

$$x^2+2 = (A_1x+1)(x^2+2x+2) + x^2(A_3x+B_3)$$

$$x^2+2 = (A_1+A_3)x^3 + (2A_1+1+B_3)x^2 + (2A_1+2)x + 2$$

Equate the coefficients on each side:

Coeff x^3	$0 = A_1+A_3$	$\rightsquigarrow$	$\boxed{A_3=1}$
" x^2	$1 = 2A_1+1+B_3$	$\rightsquigarrow$	$0=-2+B_3$ so $\boxed{B_3=2}$
" x	$0 = 2A_1+2$	$\rightsquigarrow$	$\boxed{A_1=-1}$
" 1	$2=2$ ✓		

Conclusion: $\displaystyle\int \frac{x^2+2}{x^4+2x^3+2x^2}\,dx = \int \frac{-1}{x} + \frac{1}{x^2} + \frac{x+2}{x^2+2x+2}\,dx$

$$= -\ln|x| - \frac{1}{x} + \int \frac{x+2}{x^2+2x+2}\,dx$$

Complete squares $x^2+2x+2 = (x+1)^2+1$

$$\int \frac{x+2}{x^2+2x+2}\,dx \underset{u=x+1}{\overset{}{=}} \int \frac{u+1}{1+u^2}\,du = \int \frac{u}{1+u^2}\,du + \int \frac{du}{1+u^2}$$

$$= \frac{\ln(1+u^2)}{2} + \arctan(u)$$

Answer: $-\ln|x| - \dfrac{1}{x} + \dfrac{\ln(1+(1+x)^2)}{2} + \arctan(x+1) + C.$

CONCLUSION: $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ is only as hard as $\displaystyle\int \frac{A\,dx}{(x-r)^p}$ (easy) or

$\displaystyle\int \frac{(Ax+B)\,dx}{(x^2+bx+c)^m}$ (• substitution if $A \neq 0$ $u = x^2+bx+c$
• complete squares & use trig subst if $A=0$)