Lecture XXXVIII: §10.7 Integration by parts
§10.8 A mixed bag: miscellaneous types

§1 Integration by parts:

IDEA: Reverse product rule.

$d(uv) = u\,dv + v\,du$ ⇝ integrate $uv = \int d(uv) = \int u\,dv + \int v\,du$

Write $\boxed{\int u\,dv = uv - \int v\,du}$ & pick u, v so that right integral is easier to compute than left one

Examples: (1) $\int \underbrace{x}_{u}\underbrace{e^x dx}_{dv} = xe^x - \int e^x dx = xe^x - e^x + C$

$\boxed{v = \int e^x dx = e^x}$

(2) $\int \underbrace{\ln x}_{u}\ \underbrace{dx}_{dv} = x\ln x - \int \frac{1}{x}\,x\,dx = x\ln x - \int dx = x\ln x - x + C$

$\begin{cases} v = \int dx = x \\ du = \frac{1}{x}dx \end{cases}$

(3) $\int \underbrace{(\ln x)^2}_{u}\ \underbrace{dx}_{dv} = (\ln x)^2 x - \int 2\ln x\,\underbrace{\frac{1}{x}\,x\,dx}_{=du} = x(\ln x)^2 - 2\underbrace{\int \ln x\,dx}_{\text{Ex } 2}$

$v = \int dx = x$

$= x(\ln x)^2 - 2x\ln x + 2x + C.$

Note: Integration by parts can lead to recursive formulas

Ex 1: For $n \geq 0$ integer, define $I_n = \int x^n e^x dx$

Two steps:

①. Want to write I_n in terms of $I_{n-1}, I_{n-2}, \ldots, I_0$ ⇝ parts will give this recursion

⓪. Easy case $n=0$ ("base case"): $I_0 = \int e^x dx = e^x + C$

For ①: Take $u = x^n$ $dv = e^x dx$ ⇝ $v = \int e^x dx = e^x$

$$I_n = \int x^n e^x dx = x^n e^x - \int n x^{n-1} e^x dx = x^n e^x - n I_{n-1}.$$

Repeat $(n \mapsto n-1)$ $I_n = x^n e^x - n\left(x^{n-1}e^x - (n-1) I_{n-2}\right) \overset{\text{Claim}}{=} \sum_{k=0}^{n-1} (-1)^k \frac{n!}{(n-k)!} x^{n-k} e^x + (-1)^n n! \underbrace{\boxed{I_0}}_{= e^x + C}$

[Coefficients will involve ± 1, factorials & e^x]

$\underline{A} = \left(\sum_{k=0}^{n} (-1)^n \frac{n!}{(n-k)!} x^{n-k}\right) e^x$

Ex 2: $J_p = \int \sin^p\theta \, d\theta$ for $p \geq 1$.

- Base case: $p=1$ $\quad J_1 = \int \sin\theta \, d\theta = -\cos\theta \quad (+C)$
- Recursion by parts for $p \geq 2$

$$J_p = \int \sin^p\theta \, d\theta \underset{\substack{\uparrow \\ \text{borrow} \\ \sin\theta}}{=} \int \underbrace{\sin^{p-1}\theta}_{du} \, \underbrace{\sin\theta \, d\theta}_{dv} = -\sin^{p-1}\theta \cos\theta - \int (-\cos\theta)(p-1)\sin^{p-2}\theta \cos\theta \, d\theta$$

$$v = \int \sin\theta \, d\theta = -\cos\theta$$

$$= -\sin^{p-1}\theta\cos\theta + (p-1)\int \underbrace{\cos^2\theta}_{= 1-\sin^2\theta} \sin^{p-2}\theta \, d\theta = -\sin^{p-1}\theta\cos\theta + (p-1) J_{p-2} - (p-1) J_p$$

Conclusion $\boxed{J_p} = -\sin^{p-1}\theta\cos\theta + (p-1) J_{\boxed{p-2}} - (p-1)\boxed{J_p}$

So $\quad p J_p = -\sin^{p-1}\theta\cos\theta + (p-1) J_{p-2}$

$$\boxed{J_p = -\frac{\sin^{p-1}\theta\cos\theta}{p} + \frac{(p-1)}{p} J_{p-2}}$$

p & $p-2$ have the same parity

This recursion says $\boxed{2} \to 4 \to 6 \to 8 \ldots$
$\boxed{1} \to 3 \to 5 \to 7 \ldots$ $\Big\}$ we are missing 1 base case $p=2$!

$\underline{p=2}$:

$$J_2 = \int \sin^2\theta \, d\theta \underset{\substack{\downarrow \\ \text{half-angle}}}{=} \int \frac{1-\cos 2\theta}{2} \, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} = \frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2}$$

Version 2: parts $\quad J_2 = \int \sin^2\theta \, d\theta \underset{\substack{\uparrow \\ \text{borrow} \\ \sin\theta}}{=} \int \underbrace{\sin\theta}_{=u} \, \underbrace{\sin\theta \, d\theta}_{dv} = -\sin\theta\cos\theta - \int -\cos\theta\cos\theta \, d\theta$

$$v = \int \sin\theta \, d\theta = -\cos\theta$$

$$J_2 = -\sin\theta\cos\theta + \int \underbrace{\cos^2\theta}_{1-\sin^2\theta} \, d\theta = -\sin\theta\cos\theta + \theta - J_2$$

$$\rightsquigarrow \quad J_2 = \frac{1}{2}(\theta - \sin\theta\cos\theta) \quad \checkmark$$

Obs: For p odd we could use trig integral methods from 3 lectures ago (§10.3) to get a closed formula for J_{2p+1}. For p even, our only option is the recursion above.

Claim: $\quad J_{2p+1} = \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^{k+1}\cos^{2k+1}(\theta)}{2k+1}$

Proof: $\quad \int \sin^{2p+1}\theta \, d\theta = \int \sin^{2p}\theta \, \underbrace{\sin\theta \, d\theta}_{=-d(\cos\theta)} = -\int (1-\cos^2\theta)^p \, d\cos\theta \underset{\substack{\downarrow \\ u=\cos\theta}}{=} -\int (1-u^2)^p \, du$

$$\underset{\substack{\uparrow\\ \text{Binomial Thm}}}{=} -\int \sum_{k=0}^{p} \binom{p}{k} (-1)^k u^{2k}\, du = \sum_{k=0}^{p} \binom{p}{k} (-1)^{k+1} \int u^{2k}\, du$$

$$= \sum_{k=0}^{p} \binom{p}{k} (-1)^{k+1} \frac{u^{2k+1}}{2k+1} \underset{\substack{\uparrow\\ \text{subst}\\ \text{back}}}{=} \sum_{k=0}^{p} \binom{p}{k} (-1)^{k+1} \frac{\cos^{2k+1}(\theta)}{2k+1}$$

§2 Techniques for integration

GOAL: Reduce calculations to 15 fundamental formulas on handout via

1. Substitutions
2. Trig Subs ($x = a\sin\theta$, $x = a\tan\theta$, $x = a\sec\theta$)
   (for a^2-x^2) (for a^2+x^2) (for x^2-a^2)
3. Partial Fractions & Completing the Square
4. Integration by parts
5. Trig identities + simplifications

· Three more formulas for our handout.

(16) $[a\neq 0]$
$$\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(x-a)(x+a)} = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$
$$= \frac{1}{2a}\left(\ln(x-a) - \ln(x+a)\right) + C = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + C$$

(17)
$$\int \frac{dx}{a^2-x^2} = -\int \frac{dx}{x^2-a^2} = -\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + C = \frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C$$

(18)
$$\int \frac{dx}{\sqrt{x^2+a^2}} \underset{\downarrow}{=} \int \frac{a\sec^2\theta}{a\sec\theta}\, d\theta = \int \sec\theta\, d\theta = \ln\left(\sec\theta + \tan\theta\right) + C$$
$$= \ln\left(\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right) + C$$
$$= \ln\left(\sqrt{x^2+a^2} + x\right) \underbrace{-\ln(a) + C}_{\text{const.}}$$

$x = a\tan\theta$
$dx = a\sec^2\theta\, d\theta$
$\sqrt{x^2+a^2} = a\sec\theta$

(18')
$$\int \frac{dx}{\sqrt{x^2-a^2}} \underset{\downarrow}{=} \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\, d\theta = \int \sec\theta\, d\theta = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + C$$
$$= \ln\left(x + \sqrt{x^2-a^2}\right) + \underbrace{C - \ln a}_{\text{const.}}$$

$x = a\sec\theta$
$dx = a\sec\theta\tan\theta\, d\theta$
$\sqrt{x^2-a^2} = a\tan\theta$

§3 More examples

Ex 1: $\displaystyle\int \frac{x^2}{1+x^2}\, dx \rightsquigarrow$ Long division $\displaystyle\frac{x^2}{1+x^2} = \frac{x^2+1-1}{1+x^2} = 1 - \frac{1}{1+x^2}$ immed.

$$\int \frac{x^2}{1+x^2}\,dx = \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \arctan x + C$$

Ex 2

$$\int \frac{e^{2x}}{e^x - 1}\,dx \underset{\substack{u = e^x \\ du = e^x dx}}{=} \int \frac{u\,du}{u-1} = \int 1 + \frac{1}{u-1}\,du = u + \ln(u-1) + C$$

$$= e^x + \ln(e^x - 1) + C$$

Ex 3

$$\int \frac{dx}{x(\ln x)^2} \underset{\substack{u = \ln x \\ du = \frac{dx}{x}}}{=} \int \frac{du}{u^2} = -u^{-1} + C = \frac{-1}{\ln x} + C$$

Ex 4

$$\int \frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\,\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$\underset{\substack{u = 1-x^2 \\ du = -2x\,dx}}{=} \arcsin(x) - \frac{1}{2}\int \frac{du}{u^{1/2}} = \sin^{-1} x - \sqrt{1-x^2} + C$$

Ex 5

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1}{1+\cos x}\,\frac{1-\cos x}{1-\cos x}\,dx = \int \frac{1-\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx$$

$$- \int \frac{\cos x}{\sin^2 x}\,dx = \int \underbrace{\csc^2(x)\,dx}_{d(-\cot(x))} + \frac{1}{\sin x} = \frac{1}{\sin x} - \cot(x) + C$$

Ex 6:

$$\int e^{\sqrt{x}}\,dx \underset{\substack{u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}dx = \frac{dx}{2u}}}{=} \int e^u\,2u\,du = 2\int u \underbrace{e^u du}_{dv} \underset{v = e^u}{=} 2\Big(u e^u - \underbrace{\int e^u du}_{= e^u}\Big)$$

$$= 2\left(e^{\sqrt{x}}\right)\left(\sqrt{x} - 1\right) + C$$

Ex 7: $\displaystyle\int \frac{dx}{(x^2+a^2)^n}$ for $n \geq 2$ integer (for $n=1$ use $x = a\tan x$. substitution)

Method 1: Partial fractions after integration by parts to reduce the multiplicity

$$\int \frac{dx}{(x^2+a^2)^n} = \int \frac{1}{2x}\,\underbrace{\frac{2x}{(x^2+a^2)^n}\,dx}_{dv} \underset{v = \frac{1}{-(n-1)}\frac{1}{(x^2+a^2)^{n-1}}}{=} \frac{1}{2x}\,\frac{1}{-(n-1)}\,\frac{1}{(x^2+a^2)^{n-1}} - \int \underbrace{\frac{-1}{2x^2}}_{du}\,\frac{+1}{-n+1}\,\frac{dx}{(x^2+a^2)^{n-1}}$$

$$= \frac{-1}{(n-1)\, 2x\,(x^2+a^2)^{n-1}} - \frac{1}{2(n-1)} \int \frac{dx}{x^2(x^2+a^2)^{n-1}}$$

Write $\dfrac{1}{x^2(x^2+a^2)^{n-1}} = \dfrac{A_0}{x} + \dfrac{B_0}{x^2} + \displaystyle\sum_{k=1}^{n-1} \dfrac{A_k x + B_k}{(x^2+a^2)^k}$ & repeat the integration by parts whenever ~~if~~ some $A_k = 0$ . .

Method 2 : Use a Trig substitution . $x^2+a^2 \rightsquigarrow x = a\tan\theta$

$x^2+a^2 = a^2\sec^2\theta$

$dx = a\sec^2\theta\, d\theta.$

$$\int \frac{1}{(x^2+a^2)^n}\, dx = \int \frac{a \sec^2\theta\, d\theta}{a^{2n}\sec^{2n}\theta} = \frac{1}{a^{2n-1}} \int \frac{1}{\sec^{2n-2}\theta}\, d\theta = \frac{1}{a^{2n-1}} \underbrace{\int \cos^{2(n-1)}\theta\, d\theta}_{=I_{2(n-1)}}$$

$\rightsquigarrow$ $\boxed{\text{Need to solve } I_p = \int \cos^p\theta\, d\theta}$ $= \int \underbrace{\cos^{p-1}\theta}_{=u}\ \underbrace{\cos\theta\, d\theta}_{=d(\sin\theta)} = \cos^{p-1}\theta\sin\theta - \int \sin\theta\, d(\cos^{p-1}\theta)$

$$= \cos^{p-1}\theta\sin\theta - \int \sin\theta\,(p-1)\cos^{p-2}\theta\,(-\sin\theta)\, d\theta$$

$$= \cos^{p-1}\theta\sin\theta + (p-1)\int \cos^{p-2}\theta\, \underbrace{\sin^2\theta}_{1-\cos^2\theta}\, d\theta = \cos^{p-1}\theta\sin\theta + (p-1)I_{p-2} - (p-1)I_p$$

$$\rightsquigarrow \boxed{I_p = \frac{1}{p}\cos^{p-1}\theta\sin\theta + \frac{p-1}{p} I_{p-2}}$$

$\rightsquigarrow$ recursive formula to find I_p (p & $p-2$ have the same parity)

In our case $p = 2(n-1)$ is even.

Base case $I_2 = \int \cos^2\theta\, d\theta = \int \dfrac{1+\cos 2\theta}{2}\, d\theta = \dfrac{\theta}{2} + \dfrac{\sin 2\theta}{4}$ $(+C)$

Can take $C = 0$ for I_p & get $\displaystyle\int \frac{1}{(x^2+a^2)^n}\, dx = \frac{1}{a^{2n-1}} I_{2(n-1)} + \text{Const}$