Lecture XXXIX : ~~§12.1 The MVT revisited~~
§12.2 L'Hôpital Rule. The indeterminate $0/0$

§1 Introduction:

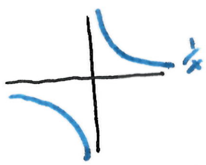
Limit Law If f, g (cnt.) functions on $x=a$, with $\lim_{x\to a} f(x) = M$ & $\lim_{x\to a} g(x) = N$

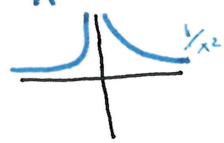
then $\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{M}{N} \quad \text{provided } N \neq 0}$

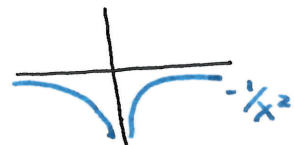
⚠ We can't use this when $N=0$. The ratio can have any behavior we want.

- If $M \neq 0$, then the limit can be $\pm\infty$ or may not exist

Ex: $\lim_{x\to 0} \frac{1}{x}$ does not exist, $\lim_{x\to 0} \frac{1}{x^2} = +\infty$, $\lim_{x\to 0} \frac{-1}{x^2} = -\infty$



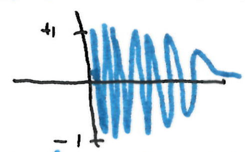




- If $M=0$, anything can happen! We have a $0/0$ indeterminacy.

Ex1: $\lim_{x\to 0} \frac{x}{x} = 1$; $\lim_{x\to 0} \frac{x^2}{x} = 0$; $\lim_{x\to 0} \pm \frac{x^2}{x^4} = \pm\infty$; $\lim_{x\to 0} \frac{x}{x^2}$ does not exist

$\lim_{x\to 0} \frac{x \sin\frac{1}{x}}{x} = \lim_{x\to 0} \sin\frac{1}{x}$ does not exists (oscillates wildly!)



Ex 2 Rational functions $\lim_{x\to a} \frac{P(x)}{Q(x)}$ P, Q polynomials (cnt)

- If a is NOT a root of $Q(x)$, limit law gives $\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$.
- If a is a root of $Q(x)$, we need to find it's multiplicity & compare it with that of $P(x)$. How?

Factor P: $P(x) = (x-a)^r \tilde{P}(x)$ with $\tilde{P}(a) \neq 0$, $r \geq 0$ integer

" Q: $Q(x) = (x-a)^l \tilde{Q}(x)$ — $\tilde{Q}(a) \neq 0$, $l \geq 1$ —"

Now $\frac{P(x)}{Q(x)} = \frac{(x-a)^r \tilde{P}(x)}{(x-a)^l \tilde{Q}(x)} = \boxed{(x-a)^{r-l}} \boxed{\frac{\tilde{P}(x)}{\tilde{Q}(x)}} \xrightarrow{x\to a} \frac{\tilde{P}(a)}{\tilde{Q}(a)}$ limit law

? ⚠ Depends on $r-l$!

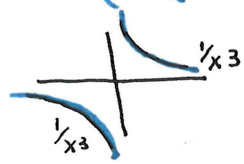
Answer for $L = \lim_{x\to a} (x-a)^{r-\ell}$ depends on the exponent.

(1) If $r-\ell = 0$, then $L = 1$

(2) If $r-\ell > 0$, then $L = 0$ $\left[\text{Eg: } (x-1)^3 \xrightarrow[x\to 1]{} 0^3 = 0\right]$

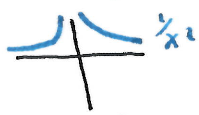
(3) If $r-\ell < 0$, then $L = +\infty$ (if $r-\ell$ even) or does not exist ($r-\ell$ odd)

Eg $a=0$, . $x^{-3} = \frac{1}{x^3}$ has no limit



vs

$x^{-2} = \frac{1}{x^2} \xrightarrow[x\to 0]{} +\infty$



Conclusion: For rational functions, we can bypass indeterminacies with algebra

Ex: $\lim_{x\to 2} \frac{3x^2-7x+2}{x^2+5x-14} = \lim_{x\to 2} \frac{(x-2)(3x-1)}{(x-2)(x+7)} = \lim_{x\to 2} \frac{(3x-1)}{(x+7)} = 5/7$

Ex 3 [Geometric soln] $\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = \operatorname{sen}'(0) = \cos(0) = 1$.

In general: $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ (if f is diff'le) is an indeterminate of the form $0/0$, so we should expect a connection between derivatives & $0/0$ indeterminates.

§ L'Hospital's Rule:

L'Hospital Thm: Fix a in $\mathbb{R}$ & pick f, g differentiable on some open interval containing a. Assume that $g'(x) \neq 0$ on this interval except perhaps at $x=a$. If $f(a) = g(a) = 0$ then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$ provided the (RHS) limit exists

Proof = later

Idea: $f(x) \approx \overset{=0}{\boxed{f(a)}} + f'(a)(x-a)$ approx is good if f' is cont at $x=a$

(linear approx) $g(x) \approx \overset{=0}{\boxed{g(a)}} + g'(a)(x-a)$ ———— g' ————

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)}{g'(a)} \frac{(x-a)}{(x-a)} = \frac{f'(a)}{g'(a)}$

This ONLY works when the approximations are good & $g'(a) \neq 0$.

In general, we'll need MVT in its general form (see later).

Examples ① $\frac{f(x)}{g(x)} = \frac{\sin(x)}{x} \xrightarrow[x\to 0]{} 1.$ (by geometry). $\frac{0}{0}$ indet.

Using L'Hospital $f_{(x)} = \sin x$ diff'ble $f'_{(x)} = \cos x$

$g(x) = 1$ —— $g'(x) = 1$ never zero around $x=0$

So $\lim_{x\to 0} \frac{\sin x}{\cancel{x}} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = \boxed{1}$

② $\frac{f(x)}{g(x)} = \frac{x^2}{\cancel{x}} \xrightarrow[x\to 0]{} x|_0 = 0$ (rat'l function) $\frac{0}{0}$ indet.

Using L'Hsp. $f(x) = x^2$ diff'ble $f'(x) = 2x$

$g(x) = x$ —— $g'(x) = 1$ never zero (around $x=0$)

So $\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} \frac{2x}{1} = 2\cdot 0 = \boxed{0}$

③ $\frac{f(x)}{g(x)} = \frac{\tan 6x}{e^{2x}-1} \longrightarrow ?$ $\frac{0}{0}$ indet.

Using L'Hospital $f(x) = \tan 6x$ diff'ble $f'(x) = 6\sec^2 6x = \frac{6}{\cos 6x}$

$g(x) = e^{2x}-1$ —— $g'(x) = 2e^{2x}$ never zero (around $x=0$)

So $\lim_{x\to 0} \frac{\tan 6x}{e^{2x}-1} = \lim_{x\to 0} \frac{\frac{6}{\cos 6x}}{2e^{2x}} = \lim_{x\to 0} \frac{3}{\cos(6x)\, e^{2x}} = \frac{3}{\cos(0)\, e^0} = \boxed{3}$

Note

We can iterate if $\frac{f'(x)}{g'(x)}$ is also an indeterminate of the form $0/0$.
(Apply L'Hsp again)

Next Time: Formal proof & other indeterminacies