Lecture XL : §12.1 The MVT revisited
§12.3 L'Hospital's Rule : Other indeterminate for

§ L'Hospital Rule for $0/0$ indeterminates:

L'Hopital Thm: Fix a in $\mathbb{R}$ & pick f, g diff'ble on some open interval $(a-\varepsilon, a+\varepsilon)$ around a. Assume that $g'(x) \neq 0$ in this interval except perhaps at $x=a$.
If $f(a)=g(a)=0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the (RHS) limit exists
[either neither of them exist, or both do & they = each other]

Moreover: If $g'(a) \neq 0$, then $(\text{RHS}) = \dfrac{f'(a)}{g'(a)}$.

Proof: later today

Q/ What if $\dfrac{f(x)}{g(x)}$ & $\dfrac{f'(x)}{g'(x)}$ are BOTH $\frac{0}{0}$ indeterminacies as $x \to a$?

A/ We iterate L'Hopital Thm! If f', g' are diff'ble around $x=a$ and $g''(x) \neq 0$ around $x=a$ (except perhaps at $x=a$), then

$$\lim_{x\to a} \underset{\frac{0}{0}}{\frac{f(x)}{g(x)}} = \lim_{x\to a} \underset{\frac{0}{0}}{\frac{f'(x)}{g'(x)}} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

In general: Apply L'Hopital until some $g^{(k)}(a) \neq 0$ & all previous $\lim\limits_{x\to a} \dfrac{f^{(j)}(x)}{g^{(j)}(x)} \sim \dfrac{0}{0}$ where $g^{(j)}(x) \neq 0$ around a except perhaps at $x=a$.

Then
$$\lim_{x\to a} \underset{\frac{0}{0}}{\frac{f(x)}{g(x)}} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \cdots = \lim_{x\to a} \underset{\frac{0}{0}}{\frac{f^{(k)}(x)}{g^{(k)}(x)}} = \frac{f^{(k)}(a)}{g^{(k)}(a)}$$

Examples ① $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} \sim \dfrac{0}{0}$

$f(x) = 1-\cos x$, $f'(x) = \operatorname{sen}(x)$
$g(x) = x^2$, $g'(x) = 2x$

$\boxed{\begin{matrix} f'(0)=0 \\ g'(0)=0 \end{matrix}}$ & $\boxed{\begin{matrix} g'(x)\neq 0 \\ \text{for } x\neq 0 \end{matrix}}$

$f''(x) = \cos(x)$ $\quad f''(0) = 1$
$g''(x) = 2$ $\quad \boxed{g''(0) = 2 \neq 0}$

$$\text{So } \lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{\{0/0 \text{ L'Hop}}{=} \lim_{x\to 0} \frac{\operatorname{sen} x}{2x} \overset{\{0/0 \text{ L'Hop.}}{=} \lim_{x\to 0} \frac{\cos(x)}{2} = \boxed{\frac{1}{2}}$$

Since the last limit exists, all previous limits also exist.

② $$\lim_{x\to 0}\frac{\operatorname{sen} x - x}{x^3} \underset{\substack{\text{L'Hop.}\\ 0/0}}{=} \lim_{x\to 0}\frac{\cos x - 1}{3x^2} \underset{\substack{\text{L'Hop}\\ 0/0}}{=} \lim_{x\to 0}\frac{-\operatorname{sen} x}{6x} \underset{\substack{\text{L'Hop}\\ 0/0}}{=} \lim_{x\to 0}\frac{-\cos x}{6} = \boxed{\frac{-1}{6}}$$

$\boxed{3x^2 \neq 0 \text{ if } x\neq 0}$ $\boxed{6x\neq 0 \text{ for } x\neq 0}$ $\boxed{6\neq 0.}$

⚠ Always verify the $0/0$ indeterminacy!

Ex: $\lim_{x\to 0}\frac{\operatorname{sen} 4x}{2x+3} = \frac{\operatorname{sen} 0}{3} = \boxed{0}$ vs $\lim_{x\to 0}\frac{(\operatorname{sen}(4x))'}{(2x+3)'} = \lim_{x\to 0}\frac{4\cos 4x}{2} = \frac{4}{2} = \boxed{2}$

$\boxed{\text{Q2: What if } x\to\pm\infty\text{ ?}}$ (ie $a=\pm\infty$). A Do a change of variables $\boxed{x=\frac{1}{t}}$

so now a becomes $\frac{1}{a}=0^+$ (for $a=\infty$) or $\frac{1}{a}=0^-$ (for $a=-\infty$)

- If $\lim_{x\to+\infty}\frac{f(x)}{g(x)} \sim \frac{0}{0}$, Then $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{t\to 0^+}\frac{f(\frac{1}{t})}{g(\frac{1}{t})}$

$$\frac{(f(\frac{1}{t}))'}{(g(\frac{1}{t}))'} = \frac{f'(\frac{1}{t})\cdot(\frac{-1}{t^2})}{g'(\frac{1}{t})\,(\frac{-1}{t^2})} = \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \frac{f'(x)}{g'(x)}.$$

(Need $g'(\frac{1}{t})\neq 0$ for $t>0$ near 0).

L'Hop Rule gives $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{t\to 0^+}\frac{f(\frac{1}{t})}{g(\frac{1}{t})} = \lim_{t\to 0^+}\frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim_{x\to+\infty}\frac{f'(x)}{g'(x)}$

If $f'(x)\neq 0$ near $+\infty$, ie for x large enough.

- Similarly for $x\to-\infty$: $\lim_{x\to-\infty}\frac{f(x)}{g(x)} = \lim_{x\to-\infty}\frac{f'(x)}{g'(x)}$ if $f'(x)\neq 0$ near $-\infty$

Conclusion: L'Hopital Rules applies also$^{0/0}$ for ~~a~~ $= \pm\infty$.

Example: $$\lim_{x\to\infty}\frac{\operatorname{sen}(\frac{1}{x})}{\frac{1}{x}} \underset{\substack{\downarrow\\ -\frac{1}{x^2}\neq 0 \text{ near } \infty}}{=} \lim_{x\to\infty}\frac{\cos(\frac{1}{x})(\frac{-1}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to\infty}\cos\frac{1}{x} = \boxed{1}$$

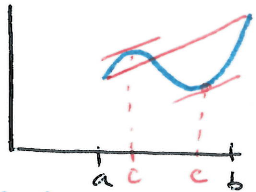
$0/0$

§2 Proof of L'Hopital Rule:

Recall: $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ is an indeterminate $0/0$ & $= f'(a)$ if this $\lim_{x\to a}$ exist

So we should expect a connection between derivatives & $0/0$ indeterminates via MVT

Mean Value Thm If $f:[a,b]\to\mathbb{R}$ cont on $[a,b]$ & diff'ble on (a,b), then there exists $a<c<b$ with $f'(c)=\dfrac{f(b)-f(a)}{b-a}$



Special case: Rolle's Thm $f(b)=f(a)$.

Generalized MVT Given $f,g:[a,b]\to\mathbb{R}$ cont on $[a,b]$ & diff'ble on (a,b), with $g'(x)\neq 0$ for all x in (a,b), then there exists $a<c<b$ with

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

(Note: $g(a)\neq g(b)$ since $g'(x)=0$ for all x in (a,b))

Proof: Consider the auxiliary function:

$$F(\textcircled{x})=(f(b)-f(a))(g(\textcircled{x})-g(a))-(g(b)-g(a))(f(\textcircled{x})-f(a))$$

- F cont on $[a,b]$, diff'ble on (a,b)
- $\left.\begin{array}{l} F(a)=0-0=0 \\ F(b)=0 \end{array}\right\}$ so $F(a)=F(b)$

Rolle's Thm ensures we can find $a<c<b$ with $F'(c)=0$

But $F'(x)=(f(b)-f(a))\,g'(x)-(g(b)-g(a))\,f'(x)$

so $F'(c)=0$ gives $\dfrac{f'(c)}{g'(c)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$. $\square$

Proof of L'Hospital Rule:

CASE 1: Assume $g'(a)\neq 0$.

Then $$\frac{f(x)}{g(x)}=\frac{f(x)-\overset{=0}{f(a)}}{\underset{\neq 0}{g(x)-g(a)}}=\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}\xrightarrow[x\to a]{}\frac{f'(a)}{g'(a)}$$

$\| \to$ Gen MVT

$\dfrac{f'(c)}{g'(c)}$ with $a<c<x$ [if $x\to a$ then $c\to a$]

Conclusion: $$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

CASE 2: $\dfrac{f(x)}{g(x)}=\dfrac{f'(c)}{g'(c)}$ with $a<c<x$ by Gen MVT

So $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{c\to a}\frac{f'(c)}{g'(c)}$ if the (RHS) limit exists.

§3 Other indeterminacies

GOAL: Extend L'Hopital Rule from $0/0$ to

$$0\cdot\infty,\quad \infty-\infty,\quad \boxed{\frac{\infty}{\infty}},\quad 0^0,\quad \infty^0,\quad 1^\infty$$

SPECIAL

via algebraic manipulations

L'Hopital Rule for $\frac{\infty}{\infty}$: Pick f, g and a diff'ble around $x=a$ with $g'(x)\neq 0$ for all $x\neq a$ near a. Assume $\lim_{x\to a}\frac{f(x)}{g(x)} \sim \frac{\infty}{\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g(x)}$$ provided the (RHS) limit exists (either in $\mathbb{R}$ or $\pm\infty$)

Proof: Say $L = \lim_{x\to a}\frac{f'(x)}{g'(x)}$. Assume $x<a$ & pick $b<x<a$.

By gen. MVT, we can find c in (b,x) with

$$\frac{f(x)-f(b)}{g(x)-g(b)} = \frac{f'(c)}{g'(c)}$$

Note: As $b\to a$, so do x & c.

Write $$\frac{f(x)-f(b)}{g(x)-g(b)} = \frac{f(x)}{g(x)}\boxed{\left(\frac{1-\frac{f(b)}{f(x)}}{1-\frac{g(b)}{g(x)}}\right)} = \frac{f'(c)}{g'(c)}$$ → can be made as close to L as we want! (if b is close enough to a.)

If b is fixed near a, then $\frac{f(b)}{f(x)} \xrightarrow[x\to a]{} \frac{f(b)}{\infty} = 0$ & $\frac{g(b)}{f(x)} \xrightarrow[x\to a]{} 0$.

So $\lim_{x\to a^-}\frac{1-\frac{f(b)}{f(x)}}{1-\frac{g(b)}{g(x)}} = 1$ so $$\lim_{x\to a^-}\frac{f(x)}{g(x)} \underset{\text{Limit Laws}}{\overset{}{=}} \lim_{x\to a^-}\frac{f(x)\left(1-\frac{f(b)}{f(x)}\right)}{g(x)\left(1-\frac{g(b)}{g(x)}\right)} = L$$ □

Same idea works for $x\to a^+$.

Remark: Similar idea works for $a=\pm\infty$

Examples: ① $$\lim_{x\to\infty}\frac{\ln x}{x^p} \underset{\substack{\infty/\infty\\ \text{L'Hop.}}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{p x^{p-1}} = \lim_{x\to\infty}\frac{1}{p x^p} = 0$$ for any $p>0$ integer

② $\lim_{x\to\infty}\frac{x^p}{e^p}$ for $p>0$ integer.

$$\| \quad \lim_{t\to 0^+}\frac{\frac{1}{t^p}}{e^{1/t}} = \lim_{t\to 0^+}\frac{-p\frac{1}{t^{p+1}}}{e^{1/t}\left(\frac{-1}{t}\right)^2} = \lim_{t\to 0^+}\frac{(-p)\frac{1}{t^{p-1}}}{e^{1/t}} = (-p)\lim_{t\to 0^+}\frac{\frac{1}{t^{p-1}}}{e^{1/t}} \quad \frac{\infty}{\infty} \text{ if } p>1$$

$$\overset{\text{repeat}}{=} (-p)(-(p-1))\overset{\infty/\infty}{\cdots}(-2)\lim_{x\to 0^+}\frac{1/t}{e^{1/t}} \overset{\infty/\infty}{=} (-1)^{p-1}p!\lim_{x\to 0^+}(-1)\frac{1}{e^{1/t}} = \boxed{0}$$